BETHE SUBALGEBRAS OF THE GROUP ALGEBRA OF THE SYMMETRIC GROUP

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Abstract. We introduce families $\mathcal{B}^S_n(z_1,\dots,z_n)$ and $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ of maximal commutative subalgebras, called Bethe subalgebras, of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group. Bethe subalgebras are deformations of the Gelfand-Zetlin subalgebra of $\mathbb{C}[\mathfrak{S}_n]$. We describe various properties of Bethe subalgebras.

1. Introduction

Algebras of integrals of motion, or Bethe algebras, play an important role in the theory of quantum integrable systems. There has been progress recently in understanding properties of Bethe algebras, see [MTV4], [FFR].

In this paper we define families $\mathcal{B}^S_n(z_1,\dots,z_n)$ and $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ of commutative subalgebras, called Bethe subalgebras, of the group algebra $\mathbb{C}[\mathfrak{S}_n]$ of the symmetric group, and describe their properties. Here $z_1,\dots,z_n$ and $\hbar\neq 0$ are complex numbers. The first family is a degeneration of the second one as $\hbar\to 0$. The Bethe subalgebras of $\mathbb{C}[\mathfrak{S}_n]$ correspond to the Bethe algebras for the Gaudin or XXX-type quantum integrable models on tensor powers of the vector representation of the Lie algebra $\mathfrak{gl}_N$ via the Schur-Weyl duality.

The subalgebras $\mathcal{B}^S_n(z_1,\dots,z_n)$ and $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ can be viewed as deformations of the Gelfand-Zetlin subalgebra $\mathcal{G}_n\subset\mathbb{C}[\mathfrak{S}_n]$. More precisely, for distinct $z_1,\dots,z_n$, the subalgebra $\mathcal{B}^S_n(z_1,\dots,z_n)$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ of dimension $\dim\mathcal{G}_n$, and $\mathcal{B}^S_n(z_1,\dots,z_n)$ tends to $\mathcal{G}_n$ as

$$\frac{z_{a-1}-z_a}{z_{a+1}-z_a}\to 0\,,\qquad a=2,\dots,n-1\,,\tag{1.1}$$

see Theorem 4.3 and Proposition 2.4. Similarly, if $z_a-z_b\neq\hbar$ for all $1\leqslant b<a\leqslant n$, then $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ of dimension $\dim\mathcal{G}_n$, and $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ tends to $\mathcal{G}_n$ in the limit (1.1) provided in addition $\hbar/(z_1-z_2)\to 0$, see Theorem 6.3 and Proposition 5.20.

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* Supported in part by NSF grant DMS-0900984
⋆ Supported in part by NSF grant DMS-0901616
◇ Supported in part by NSF grant DMS-0555327

Example. Let $n=3$. The center $\mathcal{Z}_3$ of $\mathbb{C}[\mathfrak{S}_3]$ is spanned over $\mathbb{C}$ by the identity and the elements $\sigma_{1,2}+\sigma_{1,3}+\sigma_{2,3}$, $\sigma_{1,2}\sigma_{2,3}+\sigma_{2,3}\sigma_{1,2}$, where $\sigma_{a,b}$ is the transposition of a and b. The Gelfand-Zetlin subalgebra of $\mathbb{C}[\mathfrak{S}_3]$ is spanned by $\mathcal{Z}_3$ and $\sigma_{1,2}$. The Bethe subalgebra $\mathcal{B}^S_3(z_1,z_2,z_3)$ is spanned by $\mathcal{Z}_3$ and the element $z_1\sigma_{2,3}+z_2\sigma_{1,3}+z_3\sigma_{1,2}$. The Bethe subalgebra $\mathcal{B}^S_{3,\hbar}(z_1,z_2,z_3)$ is spanned by $\mathcal{Z}_3$ and the element $z_1\sigma_{2,3}+z_2\sigma_{1,3}+z_3\sigma_{1,2}-\hbar\sigma_{1,2}\sigma_{2,3}$. Note that all maximal commutative subalgebras of $\mathbb{C}[\mathfrak{S}_3]$ are of this form.

For distinct $z_1,\dots,z_n$, consider the *KZ elements* $H^{[n]}_1,\dots,H^{[n]}_n\in\mathbb{C}[\mathfrak{S}_n]$:

$$H^{[n]}_a = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{\sigma_{a,b}}{z_a-z_b}\,,\qquad a=1,\dots,n\,. \tag{1.2}$$

These elements pairwise commute. They are the right-hand sides of Knizhnik-Zamolodchikov (*KZ*) type equations for functions with values in $\mathbb{C}[\mathfrak{S}_n]$, see for example [C], [FV]. If $\mathfrak{S}_n$ acts on $(\mathbb{C}^N)^{\otimes n}$ by permuting the tensor factors, the images of $H^{[n]}_1,\dots,H^{[n]}_n$ become the right-hand sides of the celebrated Knizhnik-Zamolodchikov equations, see [KZ], and the Hamiltonians of the Gaudin model associated with $\mathfrak{gl}_N$, see [G].

By Theorem 4.4, see also Theorem 4.7, the elements $H^{[n]}_1,\dots,H^{[n]}_n$ generate the Bethe subalgebra $\mathcal{B}^S_n(z_1,\dots,z_n)$. This statement corresponds to the well-known result that the Gelfand-Zetlin subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ is generated by the Young-Jucys-Murphy elements $J_a=\sum_{b=1}^{a-1}\sigma_{a,b}$ for $a=2,\dots,n$. Indeed, for every $a=2,\dots,n$, the element $(z_a-z_1)H^{[n]}_a$ tends to the Young-Jucys-Murphy element J_a in the limit (1.1).

The counterpart of Theorem 4.4 for the Bethe subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is given by Theorem 6.7. It implies that for distinct $z_1,\dots,z_n$ such that $z_a-z_b\neq\hbar$ for all $a,b=1,\dots,n$, the subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is generated by the *qKZ elements* $K^{[n]}_1,\dots,K^{[n]}_n$:

$$\begin{aligned} K^{[n]}_a = {}& (z_a-z_{a-1}+\hbar\sigma_{a-1,a})\,\dots(z_a-z_1+\hbar\sigma_{1,a}) \\ &\times\,(z_a-z_n+\hbar\sigma_{a,n})\dots(z_a-z_{a+1}+\hbar\sigma_{a,a+1})\,, \end{aligned} \tag{1.3}$$

cf. (6.11). The *qKZ* elements and their images in $\operatorname{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$ under the action of $\mathfrak{S}_n$ by permuting the tensor factors are limits of the right-hand sides of difference analogues of the *KZ*-type equations [C], [FR].

The spectra of $\mathcal{B}^S_n(z_1,\dots,z_n)$ and $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ as commutative algebras admit a natural description in terms of scheme-theoretic fibers of appropriate Wronski maps, see Theorems 4.3 and 6.3. This kind of description is a refinement of the nested Bethe ansatz method developed in the theory of quantum integrable models [KR].

A special case of Bethe subalgebras $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is given by the homogeneous Bethe subalgebra $\mathcal{A}^S_n=\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_1)$. Actually, $\mathcal{A}^S_n$ does not depend on $\hbar$ and z_1. The subalgebra $\mathcal{A}^S_n$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ generated by the elements

$$G^{[n]}_k = \sum_{1\leqslant i_1<\dots<i_k\leqslant n} \sigma_{i_1,i_2}\sigma_{i_2,i_3}\dots\sigma_{i_{k-1},i_k}\,,\qquad k=2,\dots,n\,,$$

see Theorem 7.3. The permutation $\sigma_{i_1,i_2}\sigma_{i_2,i_3}\dots\sigma_{i_{k-1},i_k}$ is an increasing k-cycle $(i_1\,i_2\,\dots\,i_k)$.

Another nice set of generators of $\mathcal{A}_n^S$ is given by $\gamma_n = \sigma_{1,2}\,\sigma_{2,3}\,\ldots\,\sigma_{n-1,n} = G_n^{[n]}$ and the *local charges* $I_1^{[n]},\ldots,I_{n-2}^{[n]}$, see Theorem 7.4 and Corollary 7.5. The elements $I_k^{[n]}$ and $G_k^{[n]}$ are related via the equality of generating series: $\log\bigl(1+\sum_{k=1}^{n-1}(G_n^{[n]})^{-1}G_{n-k}^{[n]}\,u^k\bigr)=\sum_{k=1}^{\infty}I_k^{[n]}\,u^k$, see (7.6) and (7.7). It is known that $I_1^{[n]},\ldots,I_{n-2}^{[n]}$ can be written as sums of local densities independent of n, see (1.4), the proof going back to [L]. In more detail, consider the chain $\mathbb{C}[\mathfrak{S}_1]\subset\mathbb{C}[\mathfrak{S}_2]\subset\cdots\subset\mathbb{C}[\mathfrak{S}_n]\subset\cdots$, where $\mathbb{C}[\mathfrak{S}_n]$ is generated by the elements $\sigma_{a,a+1}$ for $a=1,\ldots,n-1$. Then for every k there is an element $\theta_k\in\mathfrak{S}_{k+1}$ independent of n such that

$$I_k^{[n]}=\sum_{m=0}^{n-1}\gamma_n^m\,\theta_k\,\gamma_n^{-m}\,,\qquad k=1,\ldots,n-2\,,\tag{1.4}$$

cf. (7.8); for instance, $\theta_1=\sigma_{1,2}$, $\theta_2=(\sigma_{2,3}\,\sigma_{1,2}-\sigma_{1,2}\,\sigma_{2,3}-1)/2$. Notice that the image of $I_1^{[n]}$ in $\operatorname{End}\bigl((\mathbb{C}^2)^{\otimes n}\bigr)$ under the action of $\mathfrak{S}_n$ is essentially the Hamiltonian of the celebrated *XXX* Heisenberg model, whose eigenvectors and eigenvalues were first studied in [B].

The algebra $\mathcal{A}_n^S$ is semisimple, and its action on every irreducible representation of $\mathfrak{S}_n$ has simple spectrum, see Theorem 7.1.

The group algebra of the symmetric group enters various important families of algebras. An interesting question is if there are analogues of Bethe subalgebras for other members of those families. There are indications that such analogues may exist for the Weyl groups of root systems other than of type A, though there are several gaps to be closed there. The finite Hecke algebras of type A and Birman-Wenzl-Murakami algebras probably have versions of Bethe subalgebras due to their relation to centralizer constructions and analogues of the Schur-Weyl duality. The corresponding integrable models should be the models with reflecting boundary conditions, see for example [I]. It is plausible that the Bethe subalgebras can be defined for group algebras of affine Weyl groups.

The algebra $\mathcal{B}_n^S(z_1,\ldots,z_n)$ is closely related to the center of the rational Cherednik algebra of type A_n at the critical level, see [MTV8]. We expect that the Bethe subalgebra $\mathcal{B}_{n,\hbar}^S(z_1,\ldots,z_n)$ has a similar relation to the center of the trigonometric Cherednik algebra. An interesting open question is to describe an analogue of the Bethe subalgebra related to the center of the double affine Hecke algebra.

The plan of the paper is as follows. We introduce the Bethe subalgebras $\mathcal{B}_n^S(z_1,\ldots,z_n)$ in Section 2. In Section 3, we review the definition and properties of the Bethe algebra of the Gaudin model. More sophisticated properties of $\mathcal{B}_n^S(z_1,\ldots,z_n)$ are described in Section 4. We define the Bethe subalgebras $\mathcal{B}_{n,\hbar}^S(z_1,\ldots,z_n)$ in Section 5 and study their properties in Section 6. In Section 7, we consider the homogeneous Bethe subalgebra $\mathcal{A}_n^S$. Additional technical details are given in Appendix.

The authors thank M. Nazarov and A. Vershik for useful discussions.

2. Bethe subalgebras $\mathcal{B}_n^S(z_1,\ldots,z_n)$ of $\mathbb{C}[\mathfrak{S}_n]$

Let $\mathfrak{S}_m$ be the symmetric group on m symbols. For distinct $r_1,\ldots,r_m\in\{1,\ldots,n\}$ we denote by $\pi_{r_1,\ldots,r_m}^{[n]}:\mathbb{C}[\mathfrak{S}_m]\to\mathbb{C}[\mathfrak{S}_n]$ the embedding induced by the correspondence $i\mapsto r_i$.

Let $A^{[m]} = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\sigma$ be the antisymmetrizer in $\mathbb{C}[\mathfrak{S}_m]$; in particular, $A^{[1]} = 1$. Given complex numbers $z_1,\dots,z_n$, consider the polynomials $\varPhi^{[n]}_1(u),\dots,\varPhi^{[n]}_n(u)$ in one variable with coefficients in $\mathbb{C}[\mathfrak{S}_n]$:

$$\varPhi^{[n]}_i(u) \,=\, \sum_{1\leqslant r_1<\dots<r_i\leqslant n} i!\ \pi^{[n]}_{r_1,\dots,r_i}(A^{[i]}) \prod_{\substack{a=1\\ a\notin\{r_1,\dots,r_i\}}}^{n} (u-z_a) \,=\, \sum_{j=0}^{n-i}\varPhi^{[n]}_{i,j}\,u^{n-i-j}\,. \tag{2.1}$$

For instance, $\varPhi^{[n]}_1(u) = \sum_{r=1}^n\prod_{a\neq r}(u-z_a)$ and

$$\varPhi^{[n]}_{i,0} \,=\, \sum_{1\leqslant r_1<\dots<r_i\leqslant n}\pi^{[n]}_{r_1,\dots,r_i}(A^{[i]})\,,\qquad i=1,\dots,n\,. \tag{2.2}$$

Proposition 2.1 ([OV])**.** *The elements $\varPhi^{[n]}_{1,0},\dots,\varPhi^{[n]}_{n,0}$ generate the center of $\mathbb{C}[\mathfrak{S}_n]$.*

Independently, this statement follows from Proposition 3.5.

Denote by $\mathcal{B}^S_n(z_1,\dots,z_n)$ the subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ generated by all $\varPhi^{[n]}_{i,j}$, for $i=1,\dots,n$, $j=0,\dots,n-i$. The subalgebra $\mathcal{B}^S_n(z_1,\dots,z_n)$ depends on $z_1,\dots,z_n$ as parameters. Clearly,

$$\mathcal{B}^S_n(z_1,\dots,z_n) \,=\, \sigma\,\mathcal{B}^S_n(z_{\sigma(1)},\dots,z_{\sigma(n)})\,\sigma^{-1} \tag{2.3}$$

for any $\sigma\in\mathfrak{S}_n$. We call the subalgebras $\mathcal{B}^S_n(z_1,\dots,z_n)$ *Bethe subalgebras* of $\mathbb{C}[\mathfrak{S}_n]$ of *Gaudin type*.

Set

$$\varPhi^{[n]}(u,v) \,=\, v^n\prod_{a=1}^n(u-z_a) + \sum_{i=1}^n(-1)^i\,\varPhi^{[n]}_i(u)\,v^{n-i}\,, \tag{2.4}$$

where v is an indeterminate. It is straightforward to show that

$$\varPhi^{[n]}(u,v) \,=\, (-1)^n\sum_{\sigma\in\mathfrak{S}_n}\sigma\ \mathrm{sign}(\sigma)\prod_{\substack{b=1\\ b=\sigma(b)}}^{n}\bigl(1-v\,(u-z_b)\bigr)\,. \tag{2.5}$$

Lemma 2.2. *We have $\mathcal{B}^S_n(sz_1,\dots,sz_n) = \mathcal{B}^S_n(z_1,\dots,z_n)$ for any $s\neq 0$, and $\mathcal{B}^S_n(z_1+s,\dots,z_n+s) = \mathcal{B}^S_n(z_1,\dots,z_n)$ for any s.*

Proof. Formula (2.1) yields $\varPhi^{[n]}_i(su;sz_1,\dots,sz_n) = s^{n-i}\varPhi^{[n]}_i(u;z_1,\dots,z_n)$. Hence, $\varPhi^{[n]}_{i,j}(sz_1,\dots,sz_n) = s^j\,\varPhi^{[n]}_{i,j}(z_1,\dots,z_n)$, which proves the first claim. Similarly, the second claim follows from the equality $\varPhi^{[n]}_i(u+s;z_1+s,\dots,z_n+s) = \varPhi^{[n]}_i(u;z_1,\dots,z_n)$. □

Proposition 2.3. *The subalgebra $\mathcal{B}^S_n(z_1,\dots,z_n)$ is commutative.*

Proof. Consider the algebra $\mathcal{H} = \mathbb{C}[x_1,\dots,x_n]\otimes\bigl(\mathbb{C}[y_1,\dots,y_n]\ltimes\mathbb{C}[\mathfrak{S}_n]\bigr)$ and the polynomial in u,v

$$\hat\varPhi^{[n]}(u,v) \,=\, (-1)^n\sum_{\sigma\in\mathfrak{S}_n}\sigma\ \mathrm{sign}(\sigma)\prod_{b=\sigma(b)}\bigl(1-(u-x_b)\,(v-y_b)\bigr) \,=\, \sum_{i,j=0}^n\hat\varPhi^{[n]}_{i,j}\,u^{n-j}\,v^{n-i}$$

with coefficients in $\mathcal{H}$. It is shown in [MTV8] (see Theorem 2.5, Lemma 3.1 and Section 3.2 therein) that the elements $\hat{\Phi}^{[n]}_{i,j}$ commute with each other.

Let $\zeta : \mathcal{H} \to \mathbb{C}[\mathfrak{S}_n]$ be the homomorphism defined by the assignment

$$x_i \mapsto z_i\,, \quad y_i \mapsto 0\,, \quad \sigma \mapsto \sigma\,, \qquad i = 1, \dots, n\,, \quad \sigma \in \mathfrak{S}_n\,.$$

Then $\zeta\bigl(\hat{\Phi}^{[n]}_{i,j}\bigr) = 0$ for $j < i$ and $\zeta\bigl(\hat{\Phi}^{[n]}_{i,j}\bigr) = \Phi^{[n]}_{i,j-i}$ for $j \geqslant i \geqslant 1$, which yields the claim. $\square$

Alternatively, Proposition 2.3 follows from Theorem 3.1 and Corollary 3.3 below.

Let $\mathcal{Z}_m$ be the center of $\mathbb{C}[\mathfrak{S}_m]$. The subalgebra $\mathcal{G}_n \subset \mathbb{C}[\mathfrak{S}_n]$ generated by the images $\pi^{[n]}_{1,\dots,m}(\mathcal{Z}_m)$, $m = 1, \dots, n$, is called the Gelfand-Zetlin subalgebra of $\mathbb{C}[\mathfrak{S}_n]$. The subalgebra $\mathcal{G}_n$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$, see [OV].

Proposition 2.4. *The Bethe subalgebra $\mathcal{B}^S_n(z_1, \dots, z_n)$ tends to the Gelfand-Zetlin subalgebra $\mathcal{G}_n$ as $(z_{a-1} - z_a)/(z_a - z_{a+1}) \to 0$ for all $a = 2, \dots, n-1$.*

Proof. Without loss of generality we can assume that $z_1 = 0$, see Lemma 2.2, so we have $z_a/z_{a+1} \to 0$ for all $a = 2, \dots, n-1$. In this limit the element $(-1)^j\, \Phi^{[n]}_{i,j}\, z^{-1}_{n-j+1} \dots z^{-1}_n$ tends to $\pi^{[n]}_{1,\dots,n-j}\bigl(\Phi^{[n]}_{i-j,0}\bigr)$. Therefore, the limit of $\mathcal{B}^S_n(z_1, \dots, z_n)$ contains $\mathcal{G}_n$, see Proposition 2.1. Since $\mathcal{G}_n$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$, and $\mathcal{B}^S_n(z_1, \dots, z_n)$ is commutative for any $z_1, \dots, z_n$, the limit of $\mathcal{B}^S_n(z_1, \dots, z_n)$ coincides with $\mathcal{G}_n$. $\square$

Let $\beta \mapsto \beta^\dagger$ be the linear antiinvolution on $\mathbb{C}[\mathfrak{S}_n]$ such that $\sigma^\dagger = \sigma^{-1}$ for any $\sigma \in \mathfrak{S}_n$. Let $\beta \mapsto \beta^*$ be the semilinear antiinvolution on $\mathbb{C}[\mathfrak{S}_n]$ such that $\sigma^* = \sigma^{-1}$ for any $\sigma \in \mathfrak{S}_n$.

Proposition 2.5. *We have $\bigl(\Phi^{[n]}_{i,j}(z_1, \dots, z_n)\bigr)^\dagger = \Phi^{[n]}_{i,j}(z_1, \dots, z_n)$ and $\bigl(\Phi^{[n]}_{i,j}(z_1, \dots, z_n)\bigr)^* = \Phi^{[n]}_{i,j}(\bar{z}_1, \dots, \bar{z}_n)$ for all i, j. Here $\bar{z}_1, \dots, \bar{z}_n$ are the complex conjugates of $z_1, \dots, z_n$.* $\square$

Further properties of the subalgebras $\mathcal{B}^S_n(z_1, \dots, z_n)$ are given in Section 4.

3. Bethe algebra of the Gaudin model

Let $V = \mathbb{C}^N$. We identify elements of $\mathrm{End}(V)$ with $N \times N$ complex matrices. We also consider V as the natural vector representation of the group GL_N.

Let $E_{i,j} \in \mathrm{End}(V)$ be the matrix with only one nonzero entry equal to 1 at the intersection of the i-th row and j-th column. Consider first-order differential operators in u with $\mathrm{End}(V^{\otimes n})$-valued coefficients:

$$X_{i,j} \,=\, \delta_{i,j}\,\partial_u - \sum_{a=1}^{n} \frac{1^{\otimes(a-1)} \otimes E_{i,j} \otimes 1^{\otimes(n-a)}}{u - z_a}\,, \qquad i, j = 1, \dots, N\,,$$

where $\delta_{i,j}$ is the Kronecker symbol, and the n-th order differential operator in u

$$\mathcal{D} \,=\, \prod_{a=1}^{n} (u - z_a) \sum_{\sigma \in \mathfrak{S}_N} \mathrm{sign}(\sigma)\, X_{\sigma(1),1}\, X_{\sigma(2),2} \dots X_{\sigma(N),N}\,.$$

By Theorem 1 in [MTV6], $\mathcal{D}$ is a polynomial differential operator,

$$\mathcal{D} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} (-1)^i C_{i,j}^{[n]} u^{n-i-j} \partial_u^{N-i} .$$

Denote by $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ the subalgebra of $\mathrm{End}(V^{\otimes n})$ generated by all $C_{i,j}^{[n]}$ for $i = 1, \dots, n$, $j = 0, \dots, n-i$. The algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ depends on $z_1, \dots, z_n$ as parameters.

Theorem 3.1 ([T]). *The algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ is commutative and commutes with the action of GL_N on $V^{\otimes n}$.*

We call the algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ the *Bethe algebra for the Gaudin model* with parameters $z_1, \dots, z_n$.

Remark. The algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ is the image of the Bethe subalgebra of $U(\mathfrak{gl}_N[t])$ in the tensor product $\otimes_{a=1}^{n} V(z_a)$ of evaluation $\mathfrak{gl}_N[t]$-modules. We give more details in Section A.1.

Let the symmetric group $\mathfrak{S}_n$ act naturally on $V^{\otimes n}$ by permuting the tensor factors. Denote by $\varpi_n : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}(V^{\otimes n})$ the corresponding homomorphism. By Theorem 3.1, the algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ lies in the image of ϖ_n due to the Schur-Weyl duality.

Theorem 3.2. *We have $\varpi_n\big(\mathcal{B}_n^S(z_1, \dots, z_n)\big) = \mathcal{B}_{n,N}(z_1, \dots, z_n)$. More precisely, $\varpi_n(\Phi_{i,j}^{[n]}) = C_{i,j}^{[n]}$ for any $i = 1, \dots, n$, $j = 0, \dots, n-i$.*

Proof. Since $\sum_{i,j=1}^{N} E_{i,j} \otimes E_{j,i} \in \mathrm{End}(V^{\otimes 2})$ is the flip map, it is straightforward to see that the image of $\Phi_i^{[n]}(u)$ under ϖ_n coincides with $\sum_{j=0}^{n-i} C_{i,j}^{[n]} u^{n-i-j}$. □

Corollary 3.3. *The algebra $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ for $N \geqslant n$ is isomorphic to $\mathcal{B}_n^S(z_1, \dots, z_n)$.*

Proof. The homomorphism $\varpi_n : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}(V^{\otimes n})$ is injective for $N \geqslant n$. □

A partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ with at most m parts is a sequence of integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant 0$ and $\lambda_{m+1} = 0$. If $\sum_{i=1}^{\infty} \lambda_i = n$, we write $\boldsymbol{\lambda} \vdash n$ and say that $\boldsymbol{\lambda}$ is a partition of n.

For $\boldsymbol{\lambda} \vdash n$, let $M_{\boldsymbol{\lambda}}$ be the irreducible $\mathfrak{S}_n$-module corresponding to $\boldsymbol{\lambda}$, and $\chi_{\boldsymbol{\lambda}} \in \mathcal{Z}_n$ the respective central idempotent. By definition, $\chi_{\boldsymbol{\lambda}}$ acts as the identity on $M_{\boldsymbol{\lambda}}$ and acts as zero on any $M_{\boldsymbol{\mu}}$ for $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$. By Proposition 2.1, the algebra $\mathcal{B}_n^S(z_1, \dots, z_n)$ contains $\mathcal{Z}_n$, so $\chi_{\boldsymbol{\lambda}} \in \mathcal{B}_n^S(z_1, \dots, z_n)$. Set $\mathcal{B}_{n,\boldsymbol{\lambda}}^S(z_1, \dots, z_n) = \chi_{\boldsymbol{\lambda}} \mathcal{B}_n^S(z_1, \dots, z_n)$. The algebra $\mathcal{B}_{n,\boldsymbol{\lambda}}^S(z_1, \dots, z_n)$ is isomorphic to the image of $\mathcal{B}_n^S(z_1, \dots, z_n)$ in $\mathrm{End}(M_{\boldsymbol{\lambda}})$ by the canonical projection. Clearly, $\mathcal{B}_n^S(z_1, \dots, z_n) = \bigoplus_{\boldsymbol{\lambda} \vdash n} \mathcal{B}_{n,\boldsymbol{\lambda}}^S(z_1, \dots, z_n)$.

By the Schur-Weyl duality, we have the decomposition

$$V^{\otimes n} = \bigoplus_{\substack{\boldsymbol{\lambda} \vdash n \\ \lambda_{N+1} = 0}} L_{\boldsymbol{\lambda}} \otimes M_{\boldsymbol{\lambda}}$$

with respect to the $GL_N \times \mathfrak{S}_n$ action. Here $L_{\boldsymbol{\lambda}}$ is the irreducible representation of GL_N with highest weight $\boldsymbol{\lambda}$. By Theorem 3.1, the action of $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ on $V^{\otimes n}$ descends

to the action on each $M_{\boldsymbol\lambda}$. Denote by $\mathcal{B}_{n,N,\boldsymbol\lambda}(z_1,\dots,z_n)$ the image of $\mathcal{B}_{n,N}(z_1,\dots,z_n)$ in $\mathrm{End}(M_{\boldsymbol\lambda})$.

Corollary 3.4. *Let $\boldsymbol\lambda\vdash n$ be such that $\lambda_i=0$ for $i>N$. Then the algebra $\mathcal{B}_{n,N,\boldsymbol\lambda}(z_1,\dots,z_n)$ is isomorphic to $\mathcal{B}^S_{n,\boldsymbol\lambda}(z_1,\dots,z_n)$.*

Proof. The claim follows from Theorem 3.2. □

Proposition 3.5. *We have*

$$\sum_{i=0}^{n}(-1)^i\,\varPhi^{[n]}_{i,0}\prod_{j=i+1}^{n}(t+j)\;=\;\sum_{\boldsymbol\lambda\vdash n}\chi_{\boldsymbol\lambda}\prod_{j=1}^{n}(t-\lambda_j+j)\,,$$

where t is an indeterminate, $\varPhi^{[n]}_{0,0}=1$, other $\varPhi^{[n]}_{i,0}$ are given by (2.2), and $\boldsymbol\lambda=(\lambda_1,\lambda_2,\dots)$.

Proof. Let $e_{i,j}=\sum_{a=1}^{n}1^{\otimes(a-1)}\otimes E_{i,j}\otimes 1^{\otimes(n-a)}\in\mathrm{End}(V^{\otimes n})$. Set

$$(V^{\otimes n})^{sing}_{\boldsymbol\lambda}\;=\;\{\,v\in V^{\otimes n}\mid e_{i,i}v\,=\,\lambda_i\,v\,,\quad e_{j,k}v=0\,,\quad i=1,\dots,N\,,\quad 1\leqslant j<k\leqslant N\,\}\,.$$

The subspace $(V^{\otimes n})^{sing}_{\boldsymbol\lambda}$ is nonzero if and only if $\lambda_i=0$ for $i>N$. By the Schur-Weyl duality, a nonzero subspace $(V^{\otimes n})^{sing}_{\boldsymbol\lambda}$ is an $\mathfrak{S}_n$-submodule of $V^{\otimes n}$ isomorphic to $M_{\boldsymbol\lambda}$. Now the proposition follows from Theorem 3.2, Corollary 3.4, and the results of [MTV4], see formulae (2.11) and (2.3) therein. □

4. Further properties of the Bethe subalgebras $\mathcal{B}^S_n(z_1,\dots,z_n)$

The algebras $\mathcal{B}_{n,N,\boldsymbol\lambda}$ have been studied in [MTV4], see more details in Section A.1. Theorems 4.1 and 4.3 translate the results of [MTV4] into properties of the algebras $\mathcal{B}^S_n$ using Corollaries 3.3 and 3.4.

Theorem 4.1. *We have*

i) *For any $z_1,\dots,z_n$, the algebra $\mathcal{B}^S_n(z_1,\dots,z_n)$ is a Frobenius algebra.*
ii) *For real $z_1,\dots,z_n$, the algebra $\mathcal{B}^S_n(z_1,\dots,z_n)$ is a direct sum of the one-dimensional algebras isomorphic to $\mathbb{C}$. This assertion holds for generic complex $z_1,\dots,z_n$ as well.* □

We refer a reader to [W] for the definition and basic properties of Frobenius algebras.

Let $\mathfrak{S}_n$ act on $\mathbb{C}[y_1,\dots,y_n]$ by permuting the variables. Denote by $\deg p$ the homogeneous degree of $p\in\mathbb{C}[y_1,\dots,y_n]$: $\deg y_i=1$ for all $i=1,\dots,n$. We extend the degree to $M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n]$ trivially on the first factor. Then the $\mathfrak{S}_n$-module $M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n]$ is graded. For any $w\in(M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n])^{\mathfrak{S}_n}$ we have $\deg w\geqslant\sum_{i=1}^{n}(i-1)\,\lambda_i$, and the component of $(M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n])^{\mathfrak{S}_n}$ of degree $\sum_{i=1}^{n}(i-1)\,\lambda_i$ is one-dimensional, see [K]. Let $w_{\boldsymbol\lambda}$ be a nonzero element of $(M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n])^{\mathfrak{S}_n}$ of degree $\sum_{i=1}^{n}(i-1)\,\lambda_i$.

For a positive integer m and a partition $\boldsymbol\lambda$ with at most m parts, consider indeterminates $f_{i,j}$ with $i=1,\dots,m$ and $j=1,\dots,\lambda_i+m-i$, $j\neq\lambda_i-\lambda_s-i+s$ for $s=i+1,\dots,m$. Given in addition a collection of complex numbers $\boldsymbol a=(a_1,\dots,a_n)$, define the algebra $\mathcal{O}_{m,\boldsymbol\lambda,\boldsymbol a}$ as the quotient of $\mathbb{C}[f_{1,1},\dots,f_{m,\lambda_m}]$ by relations (4.2) described below.

Consider $f_{i,j}$ as coefficients of polynomials in one variable,

$$f_i(u) \,=\, u^{\lambda_i+m-i} + \sum_{j=1}^{\lambda_i+m-i} f_{i,j}\, u^{\lambda_i+m-i-j}\,, \tag{4.1}$$

assuming that $f_{i,\lambda_i-\lambda_s-i+s}=0$ for $s>i$, that is, the coefficient of u^{λ_s+m-s} in $f_i(u)$ equals zero. The defining relations for $\mathcal{O}_{m,\boldsymbol{\lambda},\boldsymbol{a}}$ are written as an equality of two polynomials in u:

$$\mathrm{Wr}[f_1(u),\dots,f_m(u)] \,=\, \prod_{1\leqslant i<j\leqslant m}(\lambda_j-\lambda_i+i-j)\,\Bigl(u^n+\sum_{s=1}^{n}(-1)^s\,a_s u^{n-s}\Bigr)\,, \tag{4.2}$$

where $\mathrm{Wr}[f_1(u),\dots,f_m(u)]=\det\bigl(\partial_u^{i-1}f_j(u)\bigr)_{i,j=1,\dots,m}$ is the Wronskian.

Lemma 4.2. *Let $\boldsymbol{\lambda}$ be a partition with at most m parts, and $k\geqslant m$. Then the algebras $\mathcal{O}_{k,\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mathcal{O}_{m,\boldsymbol{\lambda},\boldsymbol{a}}$ are isomorphic.*

Proof. Let $f^{\{k\}}_{i,j}$ and $f^{\{m\}}_{i,j}$ be the indeterminates used to define the algebras $\mathcal{O}_{k,\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mathcal{O}_{m,\boldsymbol{\lambda},\boldsymbol{a}}$, respectively. In both cases, the subscripts i,j run through the same sets of pairs because $\lambda_i=0$ for $i>m$. Denote by $f^{\{k\}}_i(u)$ and $f^{\{m\}}_i(u)$ the corresponding polynomials, see (4.1). Notice that $f^{\{k\}}_i(u)=u^{k-i}$ for $i=m+1,\dots,k$.

The assignment

$$f^{\{m\}}_{i,j}\;\mapsto\; f^{\{k\}}_{i,j}\prod_{s=m+1}^{k}\frac{\lambda_i+s-i-j}{\lambda_i+s-i}$$

defines an isomorphism of $\mathcal{O}_{m,\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mathcal{O}_{k,\boldsymbol{\lambda},\boldsymbol{a}}$. Indeed, the assignment means that

$$f^{\{m\}}_i(u)\;\mapsto\;\partial_u^{k-m}f^{\{k\}}_i(u)\,\prod_{s=m+1}^{k}\frac{1}{\lambda_i+s-i} \tag{4.3}$$

and

$$\mathrm{Wr}[f^{\{m\}}_1(u),\dots,f^{\{m\}}_m(u)]\;\mapsto\; C_{k,m,\boldsymbol{\lambda}}\,\mathrm{Wr}[f^{\{k\}}_1(u),\dots,f^{\{k\}}_k(u)]\,,$$

$$C_{k,m,\boldsymbol{\lambda}} \,=\, (-1)^{(k-m)(k+m-1)/2}\prod_{s=0}^{k-m-1}\frac{1}{s!}\,\prod_{i=1}^{m}\frac{(\lambda_i+m-i)!}{(\lambda_i+k-i)!}\,,$$

which yields the claim. □

Let $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}=\mathcal{O}_{n,\boldsymbol{\lambda},\boldsymbol{a}}$. Set

$$F_{\boldsymbol{\lambda},\boldsymbol{a}}(u,v) \,=\, e^{-uv}\,\mathrm{Wr}[f_1(u),\dots,f_n(u),e^{uv}]\prod_{1\leqslant i<j\leqslant m}\frac{1}{\lambda_j-\lambda_i+i-j}\,.$$

It is a polynomial in u,v with coefficients in $\mathcal{O}_{\boldsymbol{\lambda},\boldsymbol{a}}$:

$$F_{\boldsymbol{\lambda},\boldsymbol{a}}(u,v) \,=\, \sum_{i=0}^{n}\sum_{j=0}^{n-i}(-1)^i\,F_{\boldsymbol{\lambda},\boldsymbol{a},i,j}\,u^{n-i-j}\,v^{n-i}\,.$$

Further on, we identify elements of $M_{\boldsymbol{\lambda}}\otimes\mathbb{C}[y_1,\dots,y_n]$ with $M_{\boldsymbol{\lambda}}$-valued polynomials in $y_1,\dots,y_n$.

Theorem 4.3. *Let* $z_1, \ldots, z_n$ *be distinct. Then*

i) *The algebra* $\mathcal{B}^S_n(z_1, \ldots, z_n)$ *is a maximal commutative subalgebra of* $\mathbb{C}[\mathfrak{S}_n]$.

ii) *The map* $\mathcal{B}^S_n(z_1, \ldots, z_n) \to \bigoplus_{\boldsymbol{\lambda} \vdash n} M_{\boldsymbol{\lambda}}$, $X \mapsto \bigoplus_{\boldsymbol{\lambda} \vdash n} X w_{\boldsymbol{\lambda}}(z_1, \ldots, z_n)$, *is an isomorphism of the regular representation of* $\mathcal{B}^S_n(z_1, \ldots, z_n)$ *on itself and the* $\mathcal{B}^S_n(z_1, \ldots, z_n)$*-module* $\bigoplus_{\boldsymbol{\lambda} \vdash n} M_{\boldsymbol{\lambda}}$. *In particular,* $\dim \mathcal{B}^S_n(z_1, \ldots, z_n) = \sum_{\boldsymbol{\lambda} \vdash n} \dim M_{\boldsymbol{\lambda}}$.

iii) *If* $z_1, \ldots, z_n$ *are real, then the action of* $\mathcal{B}^S_n(z_1, \ldots, z_n)$ *on* $\bigoplus_{\boldsymbol{\lambda} \vdash n} M_{\boldsymbol{\lambda}}$ *is diagonalizable and has simple spectrum. This assertion holds for generic complex* $z_1, \ldots, z_n$ *as well.*

iv) *The assignment* $\chi_{\boldsymbol{\lambda}} \Phi^{[n]}_{i,j} \mapsto F_{\boldsymbol{\lambda}, \boldsymbol{a}, i, j}$ *for* $i = 1, \ldots, n$, $j = 0, \ldots, n-i$, *extends to an isomorphism of algebras* $\mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \ldots, z_n) \to \mathcal{O}_{\boldsymbol{\lambda}, \boldsymbol{a}}$. *Here* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $u^n + \sum_{s=1}^n (-1)^s a_s u^{n-s} = \prod_{i=1}^n (u - z_i)$.

Proof of Theorems 4.1, 4.3. Due to the decomposition $\mathcal{B}^S_n(z_1, \ldots, z_n) = \bigoplus_{\boldsymbol{\lambda} \vdash n} \mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \ldots, z_n)$, it suffices to verify the counterparts of the claims for the algebras $\mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \ldots, z_n)$ and the $\mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \ldots, z_n)$-modules $M_{\boldsymbol{\lambda}}$. The required statements follow from the properties of the algebras $\mathcal{B}_{n,N,\boldsymbol{\lambda}}(z_1, \ldots, z_n)$, established in [MTV4]. □

Remark. If $z_1, \ldots, z_n$ coincide at most in pairs, the algebra $\mathcal{B}^S_n(z_1, \ldots, z_n)$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$. If there is a triple of coinciding z's, then $\mathcal{B}^S_n(z_1, \ldots, z_n)$ is not a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$.

Let $\sigma_{a,b} \in \mathfrak{S}_n$ denote the transposition of a and b. For distinct $z_1, \ldots, z_n$, set

$$H^{[n]}_a = \sum_{\substack{b=1 \\ b \neq a}}^n \frac{\sigma_{a,b}}{z_a - z_b}, \qquad a = 1, \ldots, n,$$

cf. (1.2). It is easy to see that

$$\Phi^{[n]}_2(u) = \sum_{a=1}^n \Big(-H^{[n]}_a + \sum_{b \neq a} \frac{1}{z_a - z_b} \Big) \prod_{\substack{b=1 \\ b \neq a}}^n (u - z_b).$$

Consider the diagonal matrix

$$Z = \operatorname{diag}(z_1, \ldots, z_n) \tag{4.4}$$

and the matrix

$$Q = \begin{pmatrix} h_1 & \dfrac{1}{z_1 - z_2} & \dfrac{1}{z_1 - z_3} & \ldots & \dfrac{1}{z_1 - z_n} \\ \dfrac{1}{z_2 - z_1} & h_2 & \dfrac{1}{z_2 - z_3} & \ldots & \dfrac{1}{z_2 - z_n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \dfrac{1}{z_n - z_1} & \dfrac{1}{z_n - z_2} & \dfrac{1}{z_n - z_3} & \ldots & h_n \end{pmatrix} \tag{4.5}$$

depending on new variables $h_1, \dots, h_n$. Set

$$\mathcal{P}(u, v; z_1, \dots, z_n; h_1, \dots, h_n) \,=\, \det\bigl((u - Z)\,(v - Q) - 1\bigr)\,. \tag{4.6}$$

Theorem 4.4. *Let $z_1, \dots, z_n$ be distinct. Then the subalgebra $\mathcal{B}^S_n(z_1, \dots, z_n)$ is generated by the elements $H^{[n]}_1, \dots, H^{[n]}_n$. More precisely*

$$\Phi^{[n]}(u, v) \,=\, \mathcal{P}(u, v; z_1, \dots, z_n; H^{[n]}_1, \dots, H^{[n]}_n)\,, \tag{4.7}$$

where $\Phi^{[n]}(u, v)$ is given by (2.4).

Proof. The claim follows from Theorem 3.2 and Corollary 3.3, and [MTV7, Theorem 3.2]. Note that the matrix Q here is transposed compared with its counterpart in [MTV7]. □

Combining formulae (2.5) and (4.7), we get

$$\mathcal{P}(u, v; z_1, \dots, z_n; H^{[n]}_1, \dots, H^{[n]}_n) \,=\, (-1)^n \sum_{\sigma \in \mathfrak{S}_n} \sigma \ \mathrm{sign}(\sigma) \prod_{b = \sigma(b)} \bigl(1 - v\,(u - z_b)\bigr)\,.$$

In the limit $(z_{a-1} - z_a)\big/(z_a - z_{a+1}) \to 0$ for $a = 2, \dots, n-1$, the elements $(z_b - z_1)H^{[n]}_b$ tend to the Young-Jucys-Murphy elements $J_b = \sum_{a=1}^{b-1} \sigma_{a,b}$. The elements $J_2, \dots, J_n$ generate the Gelfand-Zetlin subalgebra $\mathcal{G}_n$, see [OV]. Theorem 4.4 is the counterpart of this fact for the Bethe subalgebra $\mathcal{B}^S_n(z_1, \dots, z_n)$.

The subalgebra of $\mathbb{C}[\mathfrak{S}_n]$, generated by the elements $H^{[n]}_1, \dots, H^{[n]}_n$, and its relation to the Gelfand-Zetlin subalgebra were considered in [CFR].

For distinct $z_1, \dots, z_n$ and a partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ of n, define the algebra $\mathcal{H}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ as the quotient of $\mathbb{C}[h_1, \dots, h_n]$ by relations (4.8), (4.9) described below. Write

$$\mathcal{P}(u, v; z_1, \dots, z_n; h_1, \dots, h_n) \,=\, \sum_{i,j=0}^{n} \mathcal{P}_{i,j}(z_1, \dots, z_n; h_1, \dots, h_n)\, u^{n-j}\, v^{n-i}\,.$$

The defining relations for $\mathcal{H}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ are

$$\mathcal{P}_{i,j}(z_1, \dots, z_n; h_1, \dots, h_n) \,=\, 0\,, \qquad 0 \leqslant j < i \leqslant n\,, \tag{4.8}$$

and

$$\sum_{i=0}^{n} \mathcal{P}_{i,i}(z_1, \dots, z_n; h_1, \dots, h_n) \prod_{j=i+1}^{n} (t + j) \,=\, \prod_{j=1}^{n} (t - \lambda_j + j)\,, \tag{4.9}$$

where t is a formal variable, cf. Proposition 3.5.

Theorem 4.5. *Let $z_1, \dots, z_n$ be distinct. Then the assignment $h_a \mapsto \chi_{\boldsymbol{\lambda}} H^{[n]}_a$, $a = 1, \dots, n$, defines an isomorphism of algebras $\mathcal{H}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ and $\mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \dots, z_n)$.*

Proof. Define the algebra $\mathcal{E}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ as the quotient of $\mathbb{C}[r_1, \dots, r_n]$ by relations (4.10), (4.11), described below.

Let $g_a(v) = (v + r_a)\, e^{z_a v}$, $a = 1, \dots, n$. Let $\Delta = \prod_{1 \leqslant a < b \leqslant n} (z_b - z_a)$. Write

$$\mathrm{Wr}\big[g_1(v), \dots, g_n(v), e^{uv}\big] \,=\, \Delta\, e^{uv + \sum_{a=1}^n z_a v} \sum_{i,j=0}^{n} R_{i,j}(z_1, \dots, z_n, r_1, \dots, r_n)\, u^{n-j}\, v^{n-i}\,.$$

The defining relations for $\mathcal{E}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ are

$$R_{i,j}(z_1, \dots, z_n, r_1, \dots, r_n) \,=\, 0\,, \qquad 0 \leqslant j < i \leqslant n\,, \tag{4.10}$$

and

$$\sum_{i=0}^{n} R_{i,i}(z_1, \dots, z_n, r_1, \dots, r_n) \prod_{j=i+1}^{n} (t + j) \,=\, \prod_{j=1}^{n} (t - \lambda_j + j)\,. \tag{4.11}$$

Formulae (4.1) – (4.3) in [MTV7] and formulae (4.8) – (4.11) in this paper imply that the algebras $\mathcal{E}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ and $\mathcal{H}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ are isomorphic by the correspondence

$$h_a \,\mapsto\, -r_a - \sum_{b \neq a} \frac{1}{z_a - z_b}\,, \qquad a = 1, \dots, n\,.$$

The algebra $\mathcal{E}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ was studied in [MTV2, Section 5], see details in Section A.2. Theorem 4.5 follows from Theorem 6.12 in [MTV2], Theorem 3.1 in [MTV6], Lemma 4.3 in [MTV7], and Corollary 3.4 in this paper. □

For a partition $\boldsymbol{\lambda}$ of n, set $\Pi_{\boldsymbol{\lambda}}(v) = \prod_{i=1}^{n} \prod_{j=1}^{\lambda_i} (v + i - j)$. Notice that $\Pi_{\boldsymbol{\lambda}}(v)$ is a polynomial in v of degree n and

$$\frac{\Pi_{\boldsymbol{\lambda}}(v)}{\Pi_{\boldsymbol{\lambda}}(v+1)} \,=\, \prod_{i=1}^{n} \frac{v - \lambda_i + i}{v + i}\,. \tag{4.12}$$

$\Pi_{\boldsymbol{\lambda}}(v)$ is the product over all boxes of the Young diagram of shape $\boldsymbol{\lambda}$, the number $i - j$ in the factor $v + i - j$ being the negative of content of the (i,j)-th box of the Young diagram.

Set $\Pi(v) \,=\, \sum_{\boldsymbol{\lambda} \vdash n} \chi_{\boldsymbol{\lambda}} \Pi_{\boldsymbol{\lambda}}(v)$. It is known that

$$\Pi(v) \,=\, \prod_{i=1}^{n} (v - J_i) \,=\, (-1)^n \sum_{\sigma \in \mathfrak{S}_n} \sigma\, (-v)^{c(\sigma)} \,=\, \sum_{\sigma \in \mathfrak{S}_n} \sigma\, \mathrm{sign}(\sigma)\, v^{c(\sigma)}\,, \tag{4.13}$$

see [J]. Here $J_1 = 0$, $J_2, \dots, J_n$ are the Young-Jucys-Murphy elements, and $c(\sigma)$ is the number of σ-orbits in $\{1, \dots, n\}$.

Consider a new generating function

$$\tilde{\varPhi}^{[n]}(u,v) \,=\, \Pi(v+1) \left(\prod_{a=1}^{n} (u - z_a) \,+\, \sum_{i=1}^{n} (-u)^i\, \varPhi_i^{[n]}(u) \prod_{j=1}^{i} \frac{1}{v + j} \right), \tag{4.14}$$

where $\varPhi_1^{[n]}(u), \dots, \varPhi_n^{[n]}(u)$ are given by (2.1), cf. (2.4).

Lemma 4.6. *The function $\tilde{\varPhi}(u,v)$ is a polynomial in v of degree n.*

Proof. Since $\Pi_{\boldsymbol\lambda}(v)$ is divisible by $(v+1)\dots(v+i)$ provided $\lambda_i \neq 0$, it suffices to show that $\chi_{\boldsymbol\lambda}\varPhi_i^{[n]}(u)=0$ if $\lambda_i=0$. This follows from formula (2.1), because $\pi^{[n]}_{r_1,\dots,r_i}(A^{[i]})$ acts by zero in the $\mathfrak{S}_n$-module $M_{\boldsymbol\lambda}$ if $\lambda_i=0$, that is, $\chi_{\boldsymbol\lambda}\pi^{[n]}_{r_1,\dots,r_i}(A^{[i]})=0$. □

Set
$$\tilde\varPhi^{[n]}(u,v)\,=\,\sum_{i=0}^n\tilde\varPhi_i^{[n]}(v)\,u^{n-i}.$$

By Proposition 3.5, and formulae (4.12) and (4.14), we have

$$\tilde\varPhi_0^{[n]}(v)\,=\,\Pi(v)\,,\qquad \tilde\varPhi_n^{[n]}(v)\,=\,(-1)^n z_1\dots z_n\,\Pi(v+1)\,. \tag{4.15}$$

Let Z,Q be the matrices given by (4.4), (4.5). Set

$$\tilde{\mathcal P}(u,v;z_1,\dots,z_n;h_1,\dots,h_n)\,=\,\det\bigl((u-Z)\,(v-ZQ)-Z\bigr)\,.$$

$$\tilde{\mathcal P}_0(v;z_1,\dots,z_n;h_1,\dots,h_n)\,=\,\det(v-ZQ)\,.$$

Theorem 4.7. *Let $z_1,\dots,z_n$ be distinct. Then*

$$\tilde\varPhi^{[n]}(u,v)\,=\,\tilde{\mathcal P}(u,v;z_1,\dots,z_n;H_1^{[n]},\dots,H_n^{[n]})\,.$$

Proof. The claim follows from Theorem 3.2 and Corollary 3.3, and appropriately modified [MTV7, Theorem 3.2]. The proof of the required counterpart of [MTV7, Theorem 3.2] is similar to the proof of the original assertion in [MTV7], using the results from [MTV3]. □

Corollary 4.8. *We have* $\tilde{\mathcal P}_0(v;z_1,\dots,z_n;H_1^{[n]},\dots,H_n^{[n]})\,=\,\Pi(v)\,.$

Proof. The claim follows from each of equalities (4.15). □

In the limit $z_a/z_{a+1}\to 0$ for $a=1,\dots,n-1$, the elements $z_1H_1^{[n]},\dots,z_nH_n^{[n]}$ tend to the the Young-Jucys-Murphy elements $J_1,\dots,J_n$, the matrix Q becomes triangular, and Corollary 4.8 reproduces the first part of formula (4.13). Moreover, in this limit

$$(-1)^i\,z_{n-i+1}^{-1}\dots z_n^{-1}\,\tilde\varPhi_i^{[n]}(v)\,\to\,(v-J_1)\dots(v-J_{n-i})\,(v+1-J_{n-i+1})\dots(v+1-J_n)$$

for $i=1,\dots,n$. Recall that $J_1=0$.

For a rational function $R(v)$ denote by $\langle R(v)\rangle$ the fractional part of $R(v)$. That is, $R(v)-\langle R(v)\rangle$ is a polynomial and $\lim_{v\to\infty}\langle R(v)\rangle=0$.

For distinct $z_1,\dots,z_n$ and a partition $\boldsymbol\lambda=(\lambda_1,\dots,\lambda_n)$ of n, define the algebra $\widetilde{\mathcal H}_{\boldsymbol\lambda}(z_1,\dots,z_n)$ as the quotient of $\mathbb C[h_1,\dots,h_n]$ by relations (4.16), (4.17) described below. Write

$$\tilde{\mathcal P}(u,v;z_1,\dots,z_n;h_1,\dots,h_n)\,=\,\sum_{i=0}^n\tilde{\mathcal P}_i(v;z_1,\dots,z_n;h_1,\dots,h_n)\,u^{n-i}\,.$$

The defining relations for $\widetilde{\mathcal H}_{\boldsymbol\lambda}(z_1,\dots,z_n)$ are

$$\tilde{\mathcal P}_0(v;z_1,\dots,z_n;h_1,\dots,h_n)\,=\,\Pi_{\boldsymbol\lambda}(v)\,, \tag{4.16}$$

cf. Corollary 4.8, and

$$\left\langle \frac{\tilde{\mathcal{P}}_i(v)}{\Pi_{\boldsymbol{\lambda}}(v+1)} \prod_{j=1}^{n-i} (v+j) \right\rangle = 0\,, \qquad i = 1, \dots, n\,. \tag{4.17}$$

Conjecture 4.9. *Let $z_1, \dots, z_n$ be distinct. Then the assignment $h_a \mapsto \chi_{\boldsymbol{\lambda}} H_a^{[n]}$, $a = 1, \dots, n$, defines an isomorphism of algebras $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}(z_1, \dots, z_n)$ and $\mathcal{B}^S_{n,\boldsymbol{\lambda}}(z_1, \dots, z_n)$.*

5. Bethe subalgebras $\mathcal{B}^S_{n,\hbar}(z_1, \dots, z_n)$ of $\mathbb{C}[\mathfrak{S}_n]$

In this section we describe further generalization of the Bethe subalgebras of $\mathbb{C}[\mathfrak{S}_n]$, depending on an additional nonzero complex parameter $\hbar$.

Let p be a complex number. Define the linear map $\mathrm{Tr}^{(p)}_m : \mathbb{C}[\mathfrak{S}_{n+m}] \to \mathbb{C}[\mathfrak{S}_n]$ by the rule:

a) write an element $\sigma \in S_{n+m}$ as a product of cycles, remove all symbols j such that $j > n$ from the record of σ, and read the obtained record as an element $\tau \in \mathfrak{S}_n$;
b) let $c(\sigma)$ be the number of σ-orbits in $\{1, \dots, m+n\}$ and $c(\tau)$ the number of τ-orbits in $\{1, \dots, n\}$;
c) then, $\mathrm{Tr}^{(p)}_m(\sigma) = p^{c(\sigma)-c(\tau)}\,\tau$.

Example. Let $n = 4$, $m = 5$, $\sigma = (137)(256)(89) \in \mathfrak{S}_9$. Then $\tau = (13) \in \mathfrak{S}_4$, $c(\sigma) = 4$, $c(\tau) = 3$, and $\mathrm{Tr}^{(p)}_5(\sigma) = p\,(13) \in \mathbb{C}[\mathfrak{S}_4]$.

The definition of the map $\mathrm{Tr}^{(p)}_m$ is motivated by Proposition 5.1. Recall that $V = \mathbb{C}^N$, and $\mathfrak{S}_k$ acts on $V^{\otimes k}$ by permuting the tensor factors, the corresponding homomorphism denoted by $\varpi_k : \mathbb{C}[\mathfrak{S}_k] \to \mathrm{End}(V^{\otimes k})$. Define the linear map $\mathrm{tr}_m : \mathrm{End}(V^{\otimes(n+m)}) \to \mathrm{End}(V^{\otimes n})$ as the trace over the last m tensor factors, that is, $\mathrm{tr}_m(X \otimes Y) = X\,\mathrm{tr}(Y)$ for any $X \in \mathrm{End}(V^{\otimes n})$ and $Y \in \mathrm{End}(V^{\otimes m})$.

Proposition 5.1. *We have $\mathrm{tr}_m\bigl(\varpi_{n+m}(\sigma)\bigr) = \varpi_n\bigl(\mathrm{Tr}^{(N)}_m(\sigma)\bigr)$ for any $\sigma \in S_{n+m}$.* □

Recall that $\pi^{[n+m]}_{r_1,\dots,r_m} : \mathbb{C}[\mathfrak{S}_m] \to \mathbb{C}[\mathfrak{S}_{n+m}]$ is the embedding induced by the correspondence $i \mapsto r_i$. For brevity, we set $\vartheta_m = \pi^{[n+m]}_{n+1,\dots,n+m}$.

Lemma 5.2. *Let $r_1, \dots, r_k$ and $s_1, \dots, s_l$ be two collections of numbers from $\{1, \dots, n+m\}$, distinct within each collection, and $\{r_1, \dots, r_k\} \cap \{s_1, \dots, s_l\} \subset \{n+1, \dots, n+m\}$. Then for any $X \in \mathfrak{S}_k$ and $Y \in \mathfrak{S}_l$, we have*

$$\mathrm{Tr}^{(p)}_m\bigl(\pi^{[n+m]}_{r_1,\dots,r_k}(X)\,\pi^{[n+m]}_{s_1,\dots,s_l}(Y)\bigr) = \mathrm{Tr}^{(p)}_m\bigl(\pi^{[n+m]}_{s_1,\dots,s_l}(Y)\,\pi^{[n+m]}_{r_1,\dots,r_k}(X)\bigr)\,. \tag{5.1}$$

Proof. Since the homomorphism $\varpi_n : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}(V^{\otimes n})$ is injective for $N \geqslant n$, formula (5.1) holds for $p \in \mathbb{Z}_{\geqslant n}$ by Proposition 5.1 and properties of the trace of matrices. This implies formula (5.1) for all p because both sides depend on p polynomially, □

Recall that $A^{[m]} \in \mathbb{C}[\mathfrak{S}_m]$ is the antisymmetrizer, $A^{[m]} = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \sigma$.

Lemma 5.3. *Let $k \leqslant m$, and $X \in \mathfrak{S}_{n+k}$. Then*

$$\mathrm{Tr}^{(p)}_m\bigl(\vartheta_m(A^{[m]})\,\pi^{[n+m]}_{1,\dots,n+k}(X)\bigr) = \mathrm{Tr}^{(p)}_k\bigl(\vartheta_k(A^{[k]})\,X\bigr) \prod_{i=1}^{m-k} \frac{p-m+i}{m+1-i}\,.$$

Proof. Straightforward by induction on m. □

Recall that $\sigma_{a,b}$ is the transposition of a and b. For $m \in \mathbb{Z}_{\geqslant 0}$, consider the polynomials $T^{[n]}_m(u;p;\hbar)$ in u with coefficients in $\mathbb{C}[\mathfrak{S}_n]$ depending on p and $\hbar$ as parameters:

$$T^{[n]}_0(u;p;\hbar) \,=\, \prod_{a=1}^{n}(u-z_a)\,, \tag{5.2}$$

$$T^{[n]}_m(u;p;\hbar) \,=\, \mathrm{Tr}^{(p)}_m\Big(\vartheta_m(A^{[m]})\,\big(u-z_n+\hbar\sum_{i=1}^{m}\sigma_{n,n+i}\big)\dots\big(u-z_1+\hbar\sum_{i=1}^{m}\sigma_{1,n+i}\big)\Big)\,. \tag{5.3}$$

Set

$$T^{[n]}_m(u;p;\hbar) \,=\, \sum_{i=0}^{n} T^{[n]}_{m,i}(p;\hbar)\,u^{n-i}\,. \tag{5.4}$$

Denote by $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ the unital subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ generated by all $T^{[n]}_{m,i}(p;\hbar)$ for $m=1,\dots,n-1$, $i=1,\dots,n$, with given $p,\hbar$. The subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ depends on $z_1,\dots,z_n$ as parameters. We call the subalgebras $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ *Bethe subalgebras* of $\mathbb{C}[\mathfrak{S}_n]$ of *XXX type*.

Proposition 5.4. *The subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ does not depend on p.*

Proof. We construct below elements $S^{[n]}_{m,i}\in\mathbb{C}[\mathfrak{S}_n]$ that do not depend on p, see (5.14), and show in Proposition 5.8 that $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is generated by $S^{[n]}_{m,i}$ for $m=1,\dots,n-1$, $i=1,\dots,n-m$. □

Lemma 5.5. *We have*

$$\big(u-z_n+\hbar\sum_{i=1}^{m}\sigma_{n,n+i}\big)\dots\big(u-z_1+\hbar\sum_{i=1}^{m}\sigma_{1,n+i}\big) \,=\, \sum_{k=0}^{\min(m,n)}\ \sum_{\substack{1\leqslant r_1<\dots<r_k\leqslant n\\ 1\leqslant s_1<\dots<s_k\leqslant m\\ \tau\in\mathfrak{S}_k}} \hbar^k\,X_{\boldsymbol{r},\boldsymbol{s},\tau}(u;\hbar)\,, \tag{5.5}$$

where

$$\begin{aligned} X_{\boldsymbol{r},\boldsymbol{s},\tau}(u;\hbar) \,=\, &\overleftarrow{\prod_{r_k<b\leqslant n}}\big(u-z_a+\hbar\sum_{j=1}^{k}\sigma_{b,n+s_j}\big)\,\times \\ &\times\overleftarrow{\prod_{1\leqslant a\leqslant k}}\Big(\sigma_{r_a,n+s_{\tau(a)}}\overleftarrow{\prod_{r_{a-1}<b<r_a}}\big(u-z_b+\hbar\sum_{j=1}^{a-1}\sigma_{b,n+s_{\tau(j)}}\big)\Big)\,, \end{aligned} \tag{5.6}$$

$\overleftarrow{\prod}$ *denotes the ordered product:* $\overleftarrow{\prod\limits_{c\leqslant i\leqslant d}} x_i \,=\, x_d\,x_{d-1}\dots x_c\,,$ *and* $r_0=0$.

Proof. Expand both sides of (5.5) as a sum of monomials $y_n\,y_{n-1}\dots y_1$, where each factor y_b is either $(u-z_b)$ or $\sigma_{b,n+j}$ for some $j\in\{1,\dots,m\}$. Then one can verify by inspection that every of $(m+1)^n$ possible monomials appear exactly once in each side of (5.5). □

Set $S_0^{[n]}(u;\hbar)=\prod_{a=1}^{n}(u-z_a)$,

$$S_k^{[n]}(u;\hbar) \,=\, k!\,\hbar^k \sum_{1\leqslant r_1<\dots<r_k\leqslant n} \pi^{[n]}_{r_1,\dots,r_k}(A^{[k]}) \prod^{\longleftarrow}_{\substack{1\leqslant a\leqslant n\\ a\notin\{r_1,\dots,r_k\}}} \Bigl(u-z_a+\hbar\sum_{\substack{j=1\\ r_j<a}}^{k}\sigma_{r_j,a}\Bigr)\,. \tag{5.7}$$

for $k=1,\dots,n,$ and $S_k^{[n]}(u;\hbar)=0$ for $k>n$. For example, since $A^{[1]}=1\,$,

$$\begin{aligned} S_1^{[n]}(u;\hbar) \,&=\, \hbar\sum_{i=1}^{n}(u-z_n+\hbar\sigma_{i,n})\dots(u-z_{i+1}+\hbar\sigma_{i,i+1})(u-z_{i-1})\dots(u-z_1) \\ &= \sum_{j=1}^{n}\,\sum_{1\leqslant i_1<\dots<i_j\leqslant n} \hbar^j\,\sigma_{i_1,i_j}\,\sigma_{i_1,i_{j-1}}\dots\sigma_{i_1,i_2}\prod_{a\notin\{i_1,\dots,i_j\}}(u-z_a)\,, \end{aligned} \tag{5.8}$$

where the empty product of transpositions for $j=1$ is the identity. Notice that the permutation $\sigma_{i_1,i_j}\,\sigma_{i_1,i_{j-1}}\dots\sigma_{i_1,i_2}=\sigma_{i_1,i_2}\,\sigma_{i_2,i_3}\dots\sigma_{i_{j-1},i_j}$ is an increasing j-cycle $(i_1\,i_2\,\dots\,i_j)$.

Proposition 5.6. *For $m\in\mathbb{Z}_{\geqslant 0}$, we have*

$$T_m^{[n]}(u;p;\hbar) \,=\, \sum_{k=0}^{m}\frac{1}{(m-k)!}\,S_k^{[n]}(u;\hbar)\prod_{i=1}^{m-k}(p-m+i)\,.$$

Proof. By formula (5.7) and Lemma 5.1, we have

$$S_k^{[n]}(u;\hbar) \,=\, k!\,\hbar^k\sum_{1\leqslant r_1<\dots<r_k\leqslant n}\mathrm{Tr}_k^{(p)}\bigl(\vartheta_k(A^{[k]})\,X_{\boldsymbol{r},\boldsymbol{s},\tau}(u;\hbar)\bigr)\,,$$

where $X_{\boldsymbol{r},\boldsymbol{s},\tau}(u;\hbar)$ is given by (5.6), and the equality holds for every $\boldsymbol{s}$ and τ. Then the claim follows by formulae (5.3), (5.5), and Lemma 5.3. ☐

Consider the generating series

$$\hat T^{[n]}(u,x;p;\hbar) \,=\, \sum_{m=0}^{\infty}T_m^{[n]}(u;p;\hbar)\,x^m\,,\qquad \hat S^{[n]}(u,x;\hbar) \,=\, \sum_{m=0}^{\infty}S_m^{[n]}(u;\hbar)\,x^m\,. \tag{5.9}$$

Remind that $\hat S(u,x;\hbar)$ is actually a polynomial in x of degree n, because $S_m^{[n]}(u;\hbar)=0$ for $m>n$. Proposition 5.6 is equivalent to $\hat T(u,x;p;\hbar)=(1+x)^p\,\hat S\bigl(u,x/(1+x);\hbar\bigr)$. Therefore, $\hat S(u,x;\hbar)=(1-x)^p\,\hat T\bigl(u,x/(1-x);p;\hbar\bigr)$, which gives

$$S_m^{[n]}(u;\hbar) \,=\, \sum_{k=0}^{m}\frac{(-1)^{m-k}}{(m-k)!}\,T_k^{[n]}(u;p,\hbar)\prod_{i=1}^{m-k}(p-m+i)\,. \tag{5.10}$$

Remark. Notice that $(1+x)^p=\sum_{m=0}^{\infty}\mathrm{Tr}_m^{(p)}(A^{[m]})\,x^m$.

Remark. Taking $p=m-1$ in either Proposition 5.6 or formula (5.10) yields $S_m^{[n]}(u;\hbar)=T_m^{[n]}(u;m-1;\hbar)$.

Lemma 5.7. *Let $m \in \mathbb{Z}_{\geqslant n}$. Then*

$$T_m^{[n]}(u;m) \,=\, \prod_{a=1}^{n}\,(u-z_a+\hbar) \tag{5.11}$$

and $T_k^{[n]}(u;m)=0$ for $k>m$.

Proof. Since $m \geqslant n$, the homomorphism $\varpi_n : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}\big((\mathbb{C}^m)^{\otimes n}\big)$ is injective. Hence, it suffices to compute the image of $T_k^{[n]}(u;m)$ under ϖ_n. This can be done using Proposition 5.1. Since $\varpi_m(A^{[k]}) = 0$ for $k>m$, verifying that $T_k^{[n]}(u;m) = 0$ for $k>m$ is straightforward. This equality also follows from Proposition 5.6, because $S_k^{[n]}(u) = 0$ for $k>n$.

To prove formula (5.11), we observe that $\varpi_m(A^{[m]})$ is a rank-one projector and use Proposition 5.1 to get

$$\mathrm{Tr}_m^{(m)}\Big(\vartheta_m(A^{[m]})\sum_{i=1}^{m}\sigma_{a,n+i}\,X\Big) \,=\, \mathrm{Tr}_m^{(m)}\Big(\vartheta_m(A^{[m]})\,X\Big) \tag{5.12}$$

for any $a=1,\dots,n$, and any $X \in \mathfrak{S}_{n+m}$. Together with formula (5.3), this yields the claim. □

Taking $p=m$ in Proposition 5.6, we have

$$\sum_{k=0}^{n}\,S_k^{[n]}(u;\hbar) \,=\, \prod_{a=1}^{n}\,(u-z_a+\hbar)\,. \tag{5.13}$$

Write

$$S_m^{[n]}(u;\hbar) \,=\, \sum_{i=0}^{n-m}\,S_{m,i}^{[n]}\,u^{n-m-i}\,. \tag{5.14}$$

Notice that by formula (5.7),

$$S_{m,0}^{[n]} \,=\, \hbar^m\,\varPhi_{m,0}^{[n]}\,. \tag{5.15}$$

Proposition 5.8. *The subalgebra $\mathcal{B}^{s}_{n,\hbar}(z_1,\dots,z_n)$ is generated by the elements $S_{m,i}^{[n]}$ for $m=1,\dots,n-1$, $i=1,\dots,n-m$.*

Proof. By formula (5.10), the subalgebra $\mathcal{B}^{s}_{n,\hbar}(z_1,\dots,z_n)$ contains the elements $S_{m,i}^{[n]}$ for all m and i. Then the claim follows from Proposition 5.6 and formula (5.13). □

Corollary 5.9. *The subalgebra $\mathcal{B}^{s}_{n,\hbar}(z_1,\dots,z_n)$ contains the elements $T_{m,i}^{[n]}$, see* (5.4), *for all m and i.*

Proof. The claim follows from Proposition 5.6. □

Lemma 5.10. *We have $\mathcal{B}^{s}_{n,s\hbar}(sz_1,\dots,sz_n) = \mathcal{B}^{s}_{n,\hbar}(z_1,\dots,z_n)$ for any $s \neq 0$, and $\mathcal{B}^{s}_{n,\hbar}(z_1+s,\dots,z_n+s) = \mathcal{B}^{s}_{n,\hbar}(z_1,\dots,z_n)$ for any s.*

Proof. Formula (5.3) yields $T_m^{[n]}(su;p;s\hbar;sz_1,\dots,sz_n) = s^n\,T_m^{[n]}(u;p;\hbar;z_1,\dots,z_n)$. Hence, $T_{m,i}^{[n]}(p;s\hbar;sz_1,\dots,sz_n) = s^i\,T_{m,i}^{[n]}(p;\hbar;z_1,\dots,z_n)$, which proves the first claim. Similarly, the second claim follows from the equality $T_m^{[n]}(u+s;p;\hbar;z_1+s,\dots,z_n+s) = T_m^{[n]}(u;p;\hbar;z_1,\dots,z_n)$. □

Let $\gamma \in \mathfrak{S}_n$ be given by $\gamma(i) = i+1$ for $i = 1, \ldots, n-1$, and $\gamma(n) = 1$.

Proposition 5.11. *We have* $\gamma\, T^{[n]}_m(u;p;\hbar;z_2,\ldots,z_n,z_1)\,\gamma^{-1} = T^{[n]}_m(u;p;\hbar;z_1,\ldots,z_n)$.

Proof. By formula (5.3) and Lemma 5.2, we have

$$
\begin{aligned}
&T^{[n]}_m(u;p;\hbar;z_1,\ldots,z_n) = \\
&= \operatorname{Tr}^{(p)}_m\Bigl(\bigl(u - z_1 + \hbar\sum_{i=1}^m \sigma_{1,n+i}\bigr)\,\vartheta_m(A^{[m]})\,\bigl(u - z_n + \hbar\sum_{i=1}^m \sigma_{n,n+i}\bigr)\ldots\bigl(u - z_2 + \hbar\sum_{i=1}^m \sigma_{2,n+i}\bigr)\Bigr) = \\
&= \operatorname{Tr}^{(p)}_m\Bigl(\vartheta_m(A^{[m]})\,\bigl(u - z_1 + \hbar\sum_{i=1}^m \sigma_{1,n+i}\bigr)\,\bigl(u - z_n + \hbar\sum_{i=1}^m \sigma_{n,n+i}\bigr)\ldots\bigl(u - z_2 + \hbar\sum_{i=1}^m \sigma_{2,n+i}\bigr)\Bigr) = \\
&= \gamma\, T^{[n]}_m(u;p;\hbar;z_2,\ldots,z_n,z_1)\,\gamma^{-1}\,.
\end{aligned}
$$

In the second equality we use that $\sum_{i=1}^m \sigma_{1,n+i}\,\vartheta_m(A^{[m]}) = \vartheta_m(A^{[m]})\sum_{i=1}^m \sigma_{1,n+i}$. □

Corollary 5.12. *We have* $\mathcal{B}^S_{n,\hbar}(z_2,\ldots,z_n,z_1) = \gamma^{-1}\,\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)\,\gamma$.

Proposition 5.13. *For any* $a = 1,\ldots,n$, *we have*

$$
\begin{aligned}
&\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,T^{[n]}_m(u;p;\hbar;z_1,\ldots,z_n) = \\
&\qquad = T^{[n]}_m(u;p;\hbar;z_1,\ldots,z_{a+1},z_a,\ldots,z_n)\,\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,.
\end{aligned}
$$

Proof. The claim follows from the identity

$$
\begin{aligned}
&\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,\Bigl(u - z_{a+1} + \hbar\sum_{i=1}^m \sigma_{a+1,n+i}\bigr)\Bigr)\,\Bigl(u - z_a + \hbar\sum_{i=1}^m \sigma_{a,n+i}\bigr)\Bigr) = \\
&= \Bigl(u - z_a + \hbar\sum_{i=1}^m \sigma_{a+1,n+i}\bigr)\Bigr)\,\Bigl(u - z_{a+1} + \hbar\sum_{i=1}^m \sigma_{a,n+i}\bigr)\Bigr)\,\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,.
\end{aligned}
$$
□

Corollary 5.14. *For any* $a = 1,\ldots,n$, *we have*

$$
\begin{aligned}
&\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n) = \\
&\qquad = \mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_{a+1},z_a,\ldots,z_n)\,\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\,.
\end{aligned}
$$

Corollary 5.15. *If* $z_a \neq z_{a+1} \pm \hbar$ *then the subalgebras* $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ *and* $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_{a+1},z_a,\ldots,z_n)$ *are conjugate in* $\mathbb{C}[\mathfrak{S}_n]$, *cf.* (2.3).

Proof. Since $\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} + \hbar\bigr)\bigl((z_a - z_{a+1})\,\sigma_{a,a+1} - \hbar\bigr) = (z_a - z_{a+1})^2 - \hbar^2 \neq 0$, the claim follows from Corollary 5.14. □

Proposition 5.16. *The subalgebra* $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ *contains the center of* $\mathbb{C}[\mathfrak{S}_n]$.

Proof. The statement follows from formula (5.15) and Proposition 2.1. □

Lemma 5.17. *We have*

$$\vartheta_m(A^{[m]})\bigl(u-(m-1)\hbar+\hbar\sigma_{a,n+1}\bigr)\,\ldots\,\bigl(u+\hbar\sigma_{a,n+1}\bigr)\;=\;\vartheta_m(A^{[m]})\bigl(u+\hbar\sum_{i=1}^{m}\sigma_{a,n+i}\bigr)\,\prod_{i=1}^{m-1}(u-i\hbar)\,.$$

Proof. Both sides of the formula are polynomials in u of degree m with the matching coefficients for u^m and u^{m-1}. In addition, the product $\prod_{i=1}^{m-1}(u-i\hbar)$ divides the left-hand side of the formula due to the identity

$$(1-\sigma_{i,j})(u-\hbar+\hbar\sigma_{a,i})(u+\hbar\sigma_{a,j})\;=\;(u-\hbar)(1-\sigma_{i,j})(u+\hbar\sigma_{a,i}+\hbar\sigma_{a,j})\,,$$

which completes the proof. □

Theorem 5.18. *The subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ is commutative.*

Proof. Let $P(u)=\prod_{a=1}^{n}(u-z_a)$. Since $\vartheta_m(A^{[m]})$ commutes with $\bigl(u+\hbar\sum_{i=1}^{m}\sigma_{a,n+i}\bigr)$, formula (5.3) can be written as

$$T^{[n]}_m(u;p;\hbar)\;=\;P(u)\;\mathrm{Tr}^{(p)}_m\biggl(\vartheta_m(A^{[m]})\Bigl(1+\frac{\hbar\sigma_{n,n+1}}{u-z_n-(m-1)\hbar}\Bigr)\,\ldots\,\Bigl(1+\frac{\hbar\sigma_{n,n+m}}{u-z_n}\Bigr)\,\ldots\,\Bigl(1+\frac{\hbar\sigma_{1,n+1}}{u-z_1-(m-1)\hbar}\Bigr)\,\ldots\,\Bigl(1+\frac{\hbar\sigma_{1,n+m}}{u-z_1}\Bigr)\biggr)\,,$$

see Lemma 5.17. Hence, for $N\in\mathbb{Z}_{>0}$, the image of $T^{[n]}_m(\hbar u;N;\hbar)/P(\hbar u)$ under the homomorphism $\varpi_n:\mathbb{C}[\mathfrak{S}_n]\to\mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$ coincides with the image of the m-th transfer matrix $\mathcal{T}_m(u)$ for the Yangian $Y(\mathfrak{gl}_N)$, see (A.1), under the homomorphism $\psi_{z_1/\hbar,\ldots,z_n/\hbar}:Y(\mathfrak{gl}_N)\to\mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$ defined in Section A.3. Therefore,

$$\varpi\bigl(\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)\bigr)\;=\;\psi_{z_1/\hbar,\ldots,z_n/\hbar}\bigl(\mathcal{B}^Y_N\bigr)\,,$$

cf. Theorem 3.2. Here $\mathcal{B}^Y_N$ is the Bethe subalgebra of $Y(\mathfrak{gl}_N)$, see Section A.3. Since the homomorphism ϖ_n is injective for $N\geqslant n$, the theorem follows from Theorem A.2. □

Proposition 5.19. *Let $z_1,\ldots,z_n$ be distinct. Then the subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ tends to $\mathcal{B}^S_n(z_1,\ldots,z_n)$ as $\hbar\to 0$.*

Proof. By formula 5.7 and Proposition 5.8, the limit of $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ contains $\mathcal{B}^S_n(z_1,\ldots,z_n)$. Since $\mathcal{B}^S_n(z_1,\ldots,z_n)$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ for distinct $z_1,\ldots,z_n$, and $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ is commutative for any $z_1,\ldots,z_n$, the limit of $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ coincides with $\mathcal{B}^S_n(z_1,\ldots,z_n)$. □

Proposition 5.20. *The Bethe subalgebra $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ tends to the Gelfand-Zetlin subalgebra $\mathcal{G}_n$ as $\hbar/(z_1-z_2)\to 0$ and $(z_{a-1}-z_a)/(z_a-z_{a+1})\to 0$ for $a=2,\ldots,n-1$.*

Proof. The limit of $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ contains $\mathcal{G}_n$ by Lemma 5.10, and Propositions 5.19 and 2.4. Since $\mathcal{G}_n$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$, and $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ is commutative for any $z_1,\ldots,z_n$, the limit of $\mathcal{B}^S_{n,\hbar}(z_1,\ldots,z_n)$ coincides with $\mathcal{G}_n$. □

Remark. Recall that $\mathcal{Z}_m$ is the center of $\mathbb{C}[\mathfrak{S}_m]$. Similarly to (5.3), for any $X \in \mathcal{Z}_m$, set

$$T^{[n]}_{m,X}(u;p;\hbar) \,=\, \mathrm{Tr}^{(p)}_m\Big(\vartheta_m(X)\,\big(u-z_n+\hbar\sum_{i=1}^m\sigma_{n,n+i}\big)\dots\big(u-z_1+\hbar\sum_{i=1}^m\sigma_{1,n+i}\big)\Big)\,,$$

$T^{[n]}_{m,X}(u;p;\hbar) \,=\, \sum_{i=0}^n T^{[n]}_{m,X,i}(p;\hbar)\,u^{n-i}$. Then $T^{[n]}_{m,X,i}(p;\hbar) \in \mathcal{B}^{S}_{n,\hbar}(z_1,\dots,z_n)$ for all $X \in \mathcal{Z}_m$ and $i=1,\dots,n$. However, the constructed linear map $\mathcal{Z}_m \to \mathcal{B}^{S}_{n,\hbar}\otimes\mathbb{C}[u]$, $X \mapsto T^{[n]}_{m,X}(u;p;\hbar)$, is not a homomorphism of algebras.

Let $\varrho \in \mathfrak{S}_n$ be given by $\varrho(i) = n-i$ for $i=1,\dots,n$.

Lemma 5.21. *Let $N \in \mathbb{Z}_{\geqslant n}$. Then*

$$\varrho\,T^{[n]}_m(u;N;\hbar;z_n,\dots,z_1)\,\varrho \,=\, (-1)^n\,T^{[n]}_{N-m}(-u-\hbar;N;\hbar;-z_1,\dots,-z_n)$$

for all $m=0,\dots,n$.

Proof. By Lemmas 5.3 and 5.2, formula (5.3) implies that

$$T^{[n]}_k(u;p,\hbar)\,\prod_{i=1}^{m-k}\frac{p-m+i}{m+1-i} \,= \tag{5.16}$$
$$=\,\mathrm{Tr}^{(p)}_m\Big(\vartheta_m(A^{[m]})\,\big(u-z_n+\hbar\sum_{i\in K}\sigma_{n,n+i}\big)\dots\big(u-z_1+\hbar\sum_{i\in K}\sigma_{1,n+i}\big)\Big)\,,$$

where K is any k-element subset of $\{1,\dots,m\}$. Then we have

$$\frac{m!\,(N-m)!}{N!}\,\varrho\,T^{[n]}_m(u;N;\hbar;z_n,\dots,z_1)\,\varrho \,=$$
$$=\,\mathrm{Tr}^{(N)}_N\Big(\vartheta_N(A^{[N]})\,\big(u-z_1+\hbar\sum_{i=1}^m\sigma_{1,n+i}\big)\,\dots\,\big(u-z_n+\hbar\sum_{i=1}^m\sigma_{n,n+i}\big)\Big) \,=$$
$$=\,\mathrm{Tr}^{(N)}_N\Big(\vartheta_N(A^{[N]})\,\big(u-z_1+\hbar\sum_{i=1}^N\sigma_{1,n+i}-\hbar\sum_{i=m+1}^N\sigma_{1,n+i}\big)\,\times$$
$$\times\,\big(u-z_2+\hbar\sum_{i=1}^m\sigma_{2,n+i}\big)\,\dots\,\big(u-z_n+\hbar\sum_{i=1}^m\sigma_{n,n+i}\big)\Big) \,=$$
$$=\,\mathrm{Tr}^{(N)}_N\Big(\vartheta_N(A^{[N]})\,\big(u-z_1+\hbar-\hbar\sum_{i=m+1}^N\sigma_{1,n+i}\big)\,\times$$
$$\times\,\big(u-z_2+\hbar\sum_{i=1}^m\sigma_{2,n+i}\big)\,\dots\,\big(u-z_n+\hbar\sum_{i=1}^m\sigma_{n,n+i}\big)\Big) \,=$$
$$=\,\mathrm{Tr}^{(N)}_N\Big(\vartheta_N(A^{[N]})\,\big(u-z_2+\hbar\sum_{i=1}^m\sigma_{2,n+i}\big)\,\dots\,\big(u-z_n+\hbar\sum_{i=1}^m\sigma_{n,n+i}\big)\,\times$$
$$\times\,\big(u-z_1+\hbar-\hbar\sum_{i=m+1}^N\sigma_{1,n+i}\big)\Big) \,=$$

where for the third equality, we use formula (5.12). Repeating similar transformations several times, we arrive at

$$= \operatorname{Tr}_N^{(N)}\Bigl(\vartheta_N(A^{[N]})\,\bigl(u - z_n + \hbar - \hbar\sum_{i=m+1}^{N}\sigma_{n,n+i}\bigr)\,\ldots\,\bigl(u - z_1 + \hbar - \hbar\sum_{i=m+1}^{N}\sigma_{1,n+i}\bigr)\Bigr) =$$

$$= (-1)^n\,\frac{m!\,(N-m)!}{N!}\,T^{[n]}_{N-m}(-u-\hbar;N;\hbar;-z_1,\ldots,-z_n)\,,$$

the last equality following from (5.16). The proposition is proved. □

Corollary 5.22. *We have* $\mathcal{B}^s_{n,\hbar}(-z_1,\ldots,-z_n) = \varrho\,\mathcal{B}^s_{n,\hbar}(z_n,\ldots,z_1)\,\varrho$.

Recall that † and * are the linear and semilinear antiinvolutions on $\mathbb{C}[\mathfrak{S}_n]$ such that $\sigma^\dagger = \sigma^* = \sigma^{-1}$ for any $\sigma\in\mathfrak{S}_n$.

Proposition 5.23. *Let* $N\in\mathbb{Z}_{\geqslant n}$. *Then*

$$\begin{aligned}\bigl(T^{[n]}_m(u;N;\hbar;z_1,\ldots,z_n)\bigr)^\dagger &= (-1)^n\,T^{[n]}_{N-m}(-u-\hbar;N;\hbar;-z_1,\ldots,-z_n)\,,\\ \bigl(T^{[n]}_m(u;N;\hbar;z_1,\ldots,z_n)\bigr)^* &= (-1)^n\,T^{[n]}_{N-m}(-\bar u-\bar\hbar;N;\bar\hbar;-\bar z_1,\ldots,-\bar z_n)\,,\end{aligned}\tag{5.17}$$

for all $m=0,\ldots,n$. *Here* $\bar u,\bar\hbar,\bar z_1,\ldots,\bar z_n$ *are the complex conjugates of* $u,\hbar,z_1,\ldots,z_n$.

Proof. It is easy to see that

$$\operatorname{Tr}^{(p)}_k(X^\dagger) = \bigl(\operatorname{Tr}^{(p)}_k(X)\bigr)^\dagger \tag{5.18}$$

for any $X\in\mathfrak{S}_{n+k}$, where we use † for both antiinvolutions of $\mathbb{C}[\mathfrak{S}_{n+k}]$ and $\mathbb{C}[\mathfrak{S}_n]$. Since

$$\bigl(T^{[n]}_m(u;p,\hbar;z_1,\ldots,z_n)\bigr)^\dagger = \varrho\,T^{[n]}_m(u;p,\hbar;z_n,\ldots,z_1)\,\varrho\,, \tag{5.19}$$

by (5.3) and (5.18), the first equality follows from Lemma 5.21. The second equality is then straightforward. □

Corollary 5.24. *We have*

$$\bigl(\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n)\bigr)^\dagger = \mathcal{B}^s_{n,\hbar}(-z_1,\ldots,-z_n) \quad \textit{and} \quad \bigl(\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n)\bigr)^* = \mathcal{B}^s_{n,\bar\hbar}(-\bar z_1,\ldots,-\bar z_n)\,.$$

6. Further properties of the Bethe subalgebras $\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n)$

For a partition $\boldsymbol\lambda$ of n, recall that $M_{\boldsymbol\lambda}$ denotes the irreducible $\mathfrak{S}_n$-module corresponding to $\boldsymbol\lambda$, and $\chi_{\boldsymbol\lambda}\in\mathbb{C}[\mathfrak{S}_n]$ — the respective central idempotent. Set $\mathcal{B}^s_{n,\hbar,\boldsymbol\lambda}(z_1,\ldots,z_n) = \chi_{\boldsymbol\lambda}\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n)$. The algebra $\mathcal{B}^s_{n,\hbar,\boldsymbol\lambda}(z_1,\ldots,z_n)$ is isomorphic to the image of $\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n)$ in $\operatorname{End}(M_{\boldsymbol\lambda})$ by the canonical projection. Clearly, $\mathcal{B}^s_{n,\hbar}(z_1,\ldots,z_n) = \bigoplus_{\boldsymbol\lambda\vdash n}\mathcal{B}^s_{n,\hbar,\boldsymbol\lambda}(z_1,\ldots,z_n)$.

Lemma 6.1. *The assignment*

$$\sigma_{i,i+1} : q(y_1,\ldots,y_n) \mapsto q(y_1,\ldots,y_{i+1},y_i,\ldots,y_n)\,\frac{y_i-y_{i+1}+\hbar}{y_i-y_{i+1}} - \frac{\hbar\,q(y_1,\ldots,y_n)}{y_i-y_{i+1}} \tag{6.1}$$

for $i=1,\ldots,n-1$, *defines an action of* $\mathfrak{S}_n$ *on* $\mathbb{C}[y_1,\ldots,y_n]$. □

Denote by $\mathbb{C}[y_1,\dots,y_n]_\hbar$ the obtained $\mathfrak{S}_n$-module. Recall that $\deg q$ denotes the homogeneous degree of $q(y_1,\dots,y_n)$. We extend the degree to $M_{\boldsymbol{\lambda}}\otimes\mathbb{C}[y_1,\dots,y_n]_\hbar$ trivially on the first factor. Notice that action (6.1) does not increase the degree, and the leading part of (6.1) acts by permuting the variables. Therefore, for every $w\in(M_{\boldsymbol{\lambda}}\otimes\mathbb{C}[y_1,\dots,y_n]_\hbar)^{\mathfrak{S}_n}$ we have $\deg w\geqslant\sum_{i=1}^n(i-1)\,\lambda_i$, and the component of $(M_{\boldsymbol{\lambda}}\otimes\mathbb{C}[y_1,\dots,y_n]_\hbar)^{\mathfrak{S}_n}$ of degree $\sum_{i=1}^n(i-1)\,\lambda_i$ is one-dimensional, see Section 4. Let $w_{\hbar,\boldsymbol{\lambda}}$ be a nonzero element of $(M_{\boldsymbol{\lambda}}\otimes\mathbb{C}[y_1,\dots,y_n]_\hbar)^{\mathfrak{S}_n}$ of degree $\sum_{i=1}^n(i-1)\,\lambda_i$.

For a positive integer m and a partition $\boldsymbol{\lambda}$ with at most m parts, consider indeterminates $f_{i,j}$ with $i=1,\dots,m$ and $j=1,\dots,\lambda_i+m-i$, $j\neq\lambda_i-\lambda_s-i+s$ for $s=i+1,\dots,m$. Given in addition a collection of complex numbers $\boldsymbol{a}=(a_1,\dots,a_n)$, define the algebra $\mathcal{O}_{m,\boldsymbol{\lambda},\boldsymbol{a}}$ as the quotient of $\mathbb{C}[f_{1,1},\dots,f_{m,\lambda_m}]$ by relations (6.2) described below.

Consider $f_{i,j}$ as coefficients of polynomials in one variable,

$$f_i(u)\,=\,u^{\lambda_i+m-i}+\sum_{j=1}^{\lambda_i+m-i}f_{i,j}\,u^{\lambda_i+m-i-j}\,,$$

assuming that $f_{i,\lambda_i-\lambda_s-i+s}=0$. The last condition means that the coefficient of u^{λ_s+m-s} in $f_i(u)$ equals zero. The defining relations for $\mathcal{O}_{m,\hbar,\boldsymbol{\lambda},\boldsymbol{a}}$ can be written as an equality of two polynomials in u:

$$\mathrm{Wr}_\hbar[f_1(u),\dots,f_m(u)]\,=\prod_{1\leqslant i<j\leqslant m}(\lambda_j-\lambda_i+i-j)\,\Big(u^n+\sum_{s=1}^n(-1)^s\,a_su^{n-s}\Big)\,,\tag{6.2}$$

where $\mathrm{Wr}_\hbar[f_1(u),\dots,f_m(u)]\,=\,\det\big(f_j(u-\hbar(i-1))\big)_{i,j=1,\dots,m}$ is the Casorati determinant (aka the discrete Wronskian).

Lemma 6.2. *Let $\boldsymbol{\lambda}$ be a partition with at most m parts, and $k\geqslant m$. Then the algebras $\mathcal{O}_{k,\hbar,\boldsymbol{\lambda},\boldsymbol{a}}$ and $\mathcal{O}_{m,\hbar,\boldsymbol{\lambda},\boldsymbol{a}}$ are isomorphic.*

Proof. The proof is similar to the proof of Lemma 4.2. The counterpart of formula (4.3) is

$$f_i^{\{m\}}(u)\,\mapsto\,(1-e^{-\hbar\partial_u})^{k-m}f_i^{\{k\}}(u)\,\prod_{s=m+1}^k\frac{1}{\hbar\,(\lambda_i+s-i)}$$

so that

$$\mathrm{Wr}_\hbar[f_1^{\{m\}}(u),\dots,f_m^{\{m\}}(u)]\,\mapsto\,\mathrm{Wr}_\hbar[f_1^{\{k\}}(u),\dots,f_k^{\{k\}}(u)]\,\prod_{s=0}^{k-m-1}\frac{1}{s!}\,\prod_{i=1}^m\frac{(\lambda_i+m-i)!}{(\lambda_i+k-i)!}\,.$$ □

Let $\mathcal{O}_{\hbar,\boldsymbol{\lambda},\boldsymbol{a}}=\mathcal{O}_{n,\hbar,\boldsymbol{\lambda},\boldsymbol{a}}$. Set

$$F(u,v)\,=\,e^{(n\hbar-u)x}\,\mathrm{Wr}_\hbar[f_1(u),\dots,f_n(u),e^{ux}\,]\,,\qquad v=e^{\hbar x}\,.\tag{6.3}$$

It is a polynomial in u,v with coefficients in $\mathcal{O}_{\hbar,\boldsymbol{\lambda},\boldsymbol{a}}$:

$$F_{\hbar,\boldsymbol{\lambda},\boldsymbol{a}}(u,v)\,=\,\sum_{i=0}^n\sum_{m=0}^n(-1)^i\,F_{\hbar,\boldsymbol{\lambda},\boldsymbol{a},m,i}\,u^{n-i}\,v^{n-m}\,.$$

Recall that we identify elements of $M_{\boldsymbol\lambda}\otimes\mathbb{C}[y_1,\dots,y_n]$ with $M_{\boldsymbol\lambda}$-valued polynomials in $y_1,\dots,y_n$.

Theorem 6.3. *Let $z_i-z_j\neq\hbar$ for all $1\leqslant j<i\leqslant n$. Then*

i) *The algebra $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$.*

ii) *The map $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)\to\bigoplus\limits_{\boldsymbol\lambda\vdash n}M_{\boldsymbol\lambda}$, $X\mapsto\bigoplus\limits_{\boldsymbol\lambda\vdash n}Xw_{\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$, is an isomorphism of the regular representation of $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ on itself and the $\mathcal{B}^S_{n,\hbar}$-module $\bigoplus\limits_{\boldsymbol\lambda\vdash n}M_{\boldsymbol\lambda}$. In particular, $\dim\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)=\sum_{\boldsymbol\lambda\vdash n}\dim M_{\boldsymbol\lambda}$.*

iii) *The action of $\mathcal{B}^S_{n,\hbar}(z_1,\dots,z_n)$ on $\bigoplus\limits_{\boldsymbol\lambda\vdash n}M_{\boldsymbol\lambda}$ is diagonalizable and has simple spectrum if one of the following assumptions holds:*

  a) *$(z_i-z_j)/\hbar$ are real for all $i,j=1,\dots,n$, and there exists an integer m such that $(z_i-z_{i+1})/\hbar>1$ for all $i=1,\dots,n-1$, $i\neq m$;*

  b) *There exists $y\in\mathbb{C}$ such that $(z_1-y)/\hbar,\dots,(z_n-y)/\hbar$ are either real or form pairs of complex conjugated numbers, and $\bigl|\operatorname{Im}((z_i-y)/\hbar)\bigr|<1/2$ for all $i=1,\dots,n$.*

  c) *$z_1,\dots,z_n$ are generic.*

iv) *The assignment $T^{[n]}_{m,i}(n;\hbar)\mapsto F_{\hbar,\boldsymbol\lambda,\boldsymbol a,m,i}$, for $i,m=1,\dots,n$, extends to an isomorphism of algebras $\mathcal{B}^S_{n,\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)\to\mathcal{O}_{\hbar,\boldsymbol\lambda,\boldsymbol a}$. Here $T^{[n]}_{m,i}(n;\hbar)$ are given by (5.4), $\boldsymbol a=(a_1,\dots,a_n)$ and $u^n+\sum_{s=1}^n(-1)^s a_su^{n-s}=\prod_{i=1}^n(u-z_i+\hbar)$.*

v) *The algebra $\mathcal{B}^S_{n,\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$ is a Frobenius algebra.*

The proof of Theorem 6.3 relies upon the results similar to those obtained in [MTV4] with the current Lie algebra $\mathfrak{gl}_N[t]$ substituted by the Yangian $Y(\mathfrak{gl}_N)$. Details will appear elsewhere.

Conjecture 6.4. *The assertions* i), iv), v) *of Theorem 6.3 hold for all $z_1,\dots,z_n$.*

The next statement, which is similar to Theorem 2.1 in [MTV5], is a rather unexpected byproduct of Theorem 6.3.

Theorem 6.5. *Let polynomials $p_1(t),\dots,p_N(t)$ be such that the polynomial*

$$W(t)\;=\;\bigl(\sqrt{-1}\,\bigr)^{N(N-1)/2}\det\bigl(p_i\bigl(t+(N+1-2j)\sqrt{-1}\,\bigr)\bigr)_{i,j=1,\dots,N}\tag{6.4}$$

has real coefficients, and all roots of $W(t)$ lie in the strip $|\operatorname{Im}t\,|\leqslant 1$. Then the vector space $X\subset\mathbb{C}[t]$, spanned by the polynomials $p_1(t),\dots,p_N(t)$, has a basis consisting of polynomials with real coefficients.

Proof. The statement follows from assertions (ii), (iii, b), and (iv) of Theorem 6.3, formula (5.17), and Proposition 5.13, provided $\hbar=2\sqrt{-1}$. ∎

Remark. If $p_1(t),\dots,p_N(t)$ have real coefficients, then $W(t)$ has real coefficients too.

Example. Let $N=2$, $p_1(t)=t+a$, $p_2(t)=t^3+bt^2+c$, and $W(t)$ is given by (6.4). Suppose $W(t)=4t^3+At^2+Bt$, with real A and B. Then the numbers a,b,c are real if

and only if $B \leqslant 4 + A^2/12$, whereas Theorem 6.5 asserts that the numbers a, b, c are real provided $B \leqslant 4 + A^2/16$.

Consider the matrix Z given by (4.4) and the matrix $Q_\hbar$ with the entries

$$(Q_\hbar)_{ab} \,=\, \frac{c_a}{z_a - z_b + \hbar}\,, \tag{6.5}$$

where $c_1, \ldots, c_n$ are new variables. Given a polynomial $q \in \mathbb{C}[u]$, set

$$c_a \,=\, q(z_a) \prod_{\substack{b=1\\ b\neq a}}^{n} \frac{1}{z_a - z_b}\,, \qquad a = 1, \ldots, n\,, \tag{6.6}$$

and define

$$\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q) \,=\, \det\bigl((u - Z)\,(v - Q_\hbar) - \hbar Q_\hbar\bigr)\,, \tag{6.7}$$

where $Q_\hbar$ is given by (6.5), (6.6). Then $\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q)$ depends polynomially on the coefficients of $q(u)$ and rationally on $z_1, \ldots, z_n$ with possible poles at the hyperplanes $z_a = z_b$ and $z_a + \hbar = z_b$ for $a \neq b$.

Lemma 6.6. *The expression $\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q)$ is regular at the hyperplanes $z_a = z_b$.*

Proof. It follows from (6.7) and (6.6) that $\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q)$ is invariant under permutations of $z_1, \ldots, z_n$. So it suffices to show that $\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q)$ is regular at the hyperplane $z_1 = z_2$. The order of the pole in question is at most two, and it is straightforward to see that the coefficient of $(z_1 - z_2)^{-2}$ actually vanishes. Since $\mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; q)$ is invariant under the exchange of z_1 and z_2, the coefficient of $(z_1 - z_2)^{-1}$ vanishes too, which proves the lemma. □

Write

$$S_1^{[n]}(u; \hbar) \,=\, \hbar n u^{n-1} + \sum_{i=1}^{n-1} \hbar^{i+1} S_{1,i}^{[n]}\, u^{n-i-1}\,, \tag{6.8}$$

see (5.8). Recall that $T_1^{[n]}(u; p; \hbar) = S_1^{[n]}(u; \hbar) + p \prod_{a=1}^{n}(u - z_a)$, see Proposition (5.6).

Theorem 6.7. *Let $z_a - z_b \neq \hbar$ for all $a, b = 1, \ldots, n$. Then the subalgebra $\mathcal{B}^S_{n,\hbar}(z_1, \ldots, z_n)$ is generated by the elements $S_{1,1}^{[n]}, \ldots, S_{1,n-1}^{[n]}$. More precisely,*

$$T^{[n]}(u, v; \hbar) \,=\, \mathcal{P}_\hbar(u, v; z_1, \ldots, z_n; S_1^{[n]})\,, \tag{6.9}$$

where

$$T^{[n]}(u, v; \hbar) \,=\, \sum_{m=0}^{n} (-1)^m\, T_m^{[n]}(u; n; \hbar)\, v^{n-m} \,=\, \sum_{k=0}^{n} (-1)^k\, S_k^{[n]}(u; \hbar)\, (v-1)^{n-k}\,, \tag{6.10}$$

see (5.4), (5.7).

Proof. For distinct $z_1, \ldots, z_n$, the proof is similar to the proof of Theorem 3.2 in [MTV7], using the results from [MTV3]. In this case, $\mathcal{B}^S_{n,\hbar}(z_1, \ldots, z_n)$ is generated by the elements $K_a^{[n]} = T_1^{[n]}(z_a; \hbar) = S_1^{[n]}(z_a; \hbar)$ for $a = 1, \ldots, n$, and hence, by the elements $S_{1,1}^{[n]}, \ldots, S_{1,n-1}^{[n]}$, because $S_1^{[n]}(u; \hbar)$ is a polynomial in u of degree $n - 1$. Since both sides of equality (6.9) are regular at the hyperplanes $z_a = z_b$, see Lemma 6.6, the statement follows. □

Remark. We have $T^{[n]}(u,v;\hbar) = v^n\,\hat T(u,-v^{-1};n;\hbar) = (v-1)^n\,\hat S\bigl(u,(1-v)^{-1};\hbar\bigr)$, see (5.9).

Remark. Notice that

$$
\begin{aligned}
K_a^{[n]} = {} & (z_a - z_{a-1} + \hbar\sigma_{a-1,a})\,\dots(z_a - z_1 + \hbar\sigma_{1,a}) \\
& \times (z_a - z_n + \hbar\sigma_{a,n})\dots(z_a - z_{a+1} + \hbar\sigma_{a,a+1}),
\end{aligned}
\tag{6.11}
$$

cf. (1.3). We call $K_1^{[n]},\dots,K_n^{[n]}$ the *qKZ elements*. They commute with each other. Since

$$
K_1^{[n]}\dots\,K_n^{[n]} = \prod_{\substack{a,b=1\\ a\neq b}}^{n}(z_a - z_b + \hbar)\,,
$$

the *qKZ* elements are invertible if $z_a - z_b \neq \hbar$ for all $a,b = 1,\dots,n$.

For $z_1,\dots,z_n$ such that $z_a - z_b \neq \hbar$ for all a,b, and a partition $\boldsymbol\lambda = (\lambda_1,\dots,\lambda_n)$ of n, define the algebra $\mathcal{H}_{\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$ as the quotient of $\mathbb{C}[q_1,\dots,q_n]$ by relations (6.12), (6.13) described below. Consider a polynomial $q(u) = \sum_{i=1}^n q_i\,u^{n-i}$, and write

$$
\mathcal{P}_\hbar(u,v;z_1,\dots,z_n;q) = \sum_{i,j=0}^{n}\mathcal{P}_{\hbar;i,j}(z_1,\dots,z_n;q)\,u^{n-j}\,(v-1)^{n-i}\,,
$$

see (6.7). The defining relations for $\mathcal{H}_{\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$ are

$$
\mathcal{P}_{\hbar;i,j}(z_1,\dots,z_n;q) = 0\,,\qquad 0\leqslant j<i\leqslant n\,,
\tag{6.12}
$$

and

$$
\sum_{i=0}^{n}\mathcal{P}_{\hbar;i,i}(z_1,\dots,z_n;q)\prod_{j=i+1}^{n}(t+j) = \prod_{j=1}^{n}(t-\lambda_j+j)\,,
\tag{6.13}
$$

where t is a formal variable, cf. (4.8), (4.9).

Conjecture 6.8. *Let $z_a - z_b \neq \hbar$ for all $a,b = 1,\dots,n$. Then the assignment $q_1 \mapsto \hbar\chi_{\boldsymbol\lambda}$ and $q_i \mapsto \chi_{\boldsymbol\lambda} S_{1,i-1}^{[n]}$ for $i = 2,\dots,n$, defines an isomorphism of algebras $\mathcal{H}_{\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$ and $\mathcal{B}^S_{n,\hbar,\boldsymbol\lambda}(z_1,\dots,z_n)$.*

7. Homogeneous Bethe subalgebra $\mathcal{A}_n^S$ of $\mathbb{C}[\mathfrak{S}_n]$

Consider the subalgebra $\mathcal{A}_n^S = \mathcal{B}^S_{n,\hbar}(z_1,\dots,z_1) \subset \mathbb{C}[\mathfrak{S}_n]$. By Lemma 5.10, it does not depend on $\hbar$ and z_1. We call $\mathcal{A}_n^S$ the *homogeneous Bethe subalgebra* of $\mathbb{C}[\mathfrak{S}_n]$. By Theorem 6.3, the subalgebra $\mathcal{A}_n^S$ is a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$ of dimension $\sum_{\boldsymbol\lambda\vdash n}\dim M_{\boldsymbol\lambda}$. We also have $\bigl(\mathcal{A}_n^S\bigr)^\dagger = \bigl(\mathcal{A}_n^S\bigr)^* = \mathcal{A}_n^S$, see Corollary 5.24.

Further on throughout this section we assume that $\hbar = 1$ and $z_1 = \dots = z_n = 0$.

Denote by Θ the set of n-dimensional subspaces $U\subset\mathbb{C}[u]$ with the property: U has a basis $p_1,\dots,p_n$ such that

$$
\mathrm{Wr}_\hbar[p_1(u),\dots,p_n(u)] = (u+1)^n.
$$

For $U \in \Theta$ with a basis $p_1, \ldots, p_n$ as above, set

$$F_U(u,v) \,=\, e^{(n-u)x}\, \mathrm{Wr}_\hbar[p_1(u), \ldots, p_n(u), e^{ux}]\,, \qquad v = e^x\,. \tag{7.1}$$

Clearly, $F_U(u,v)$ does not depend on a choice of the basis.

For a partition $\boldsymbol{\lambda}$ of n, define the subset $\Theta_{\boldsymbol{\lambda}} \subset \Theta$ as follows: $U \in \Theta_{\boldsymbol{\lambda}}$ if U has a basis $p_1, \ldots, p_n$ such that $\deg p_i = \lambda_i + n - i\,,\ i = 1, \ldots, n$.

Theorem 7.1. *The action of $\mathcal{A}_n^S$ on $\bigoplus_{\boldsymbol{\lambda}\vdash n} M_{\boldsymbol{\lambda}}$ is diagonalizable and has simple spectrum. The eigenvectors of $\mathcal{A}_n^S$ are in a bijection with the elements of Θ. If w_U is an eigenvector of $\mathcal{A}_n^S$, corresponding to $U \in \Theta$, and $T^{[n]}(u,v)$ is given by (6.10) with $\hbar = 1$, then*

$$T^{[n]}(u,v)\, w_U \,=\, F_U(u,v)\, w_U\,,$$

The vector w_U lies in the direct summand $M_{\boldsymbol{\lambda}} \subset \bigoplus_{\boldsymbol{\lambda}\vdash n} M_{\boldsymbol{\lambda}}$ if and only if $U \in \Theta_{\boldsymbol{\lambda}}$.

Proof. The theorem follows from items iii) and iv) of Theorem 6.3. □

Remark. For $k \leqslant n$, denote by Θ_k the set of k-dimensional subspaces $U \subset \mathbb{C}[u]$ with the property: U has a basis $p_1, \ldots, p_k$ such that

$$\mathrm{Wr}_\hbar[p_1(u), \ldots, p_k(u)] \,=\, (u+1)^n\,.$$

For $U \in \Theta_k$ with a basis $p_1, \ldots, p_k$ as above, set

$$F_U(u,v) \,=\, e^{(k-u)x}\, \mathrm{Wr}_\hbar[p_1(u), \ldots, p_k(u), e^{ux}]\,, \qquad v = e^x\,.$$

Set $\Theta_{(k)} = \bigcup_{\boldsymbol{\lambda}\vdash n, \lambda_{k+1}=0} \Theta_{\boldsymbol{\lambda}}$. For $U \in \Theta_{(k)}$ with a basis $p_1, \ldots, p_n$, let $\delta_k(U) \subset \mathbb{C}[u]$ be the subspace, spanned by the polynomials $(1 - e^{-\partial_u})^{n-k} p_i(u)$, $i = 1, \ldots, n$. It is easy to see that δ_k is a bijection from $\Theta_{(k)}$ to Θ_k and $F_U(u,v) \,=\, (v-1)^{n-k} F_{\delta_k(U)}(u,v)$.

Set

$$G_k^{[n]} \,=\, \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \sigma_{i_1,i_2}\sigma_{i_2,i_3} \ldots \sigma_{i_{k-1},i_k}\,, \qquad k = 2, \ldots, n\,. \tag{7.2}$$

By formula (5.8) with $\hbar = 1$, we have

$$S_1^{[n]}(u) \,=\, n u^{n-1} + \sum_{k=2}^{n} G_k^{[n]}\, u^{n-k}\,. \tag{7.3}$$

Recall that $T_1^{[n]}(u;p;\hbar) = p\,u^n + S_1^{[n]}(u;\hbar)$.

Let $\widehat{Z}$ be the matrix with entries $\widehat{Z}_{ab} \,=\, \delta_{a,b-1}\,,\ a,b = 1, \ldots, n$. Given a polynomial $q \in \mathbb{C}[u]$, consider the matrix $\widehat{Q}$ with entries

$$\widehat{Q}_{ab} \,=\, \frac{1}{(n-a)!}\, \frac{d^{n-a}}{du^{n-a}} \Big(\frac{q(u)}{(u+1)^b} \Big)\Big|_{u=0} \tag{7.4}$$

and define

$$\widehat{\mathcal{P}}(u,v;q) \,=\, \det\big((u - \widehat{Z}\,)(v - \widehat{Q}\,) - \widehat{Q}\,\big)\,, \tag{7.5}$$

cf. (6.7). $\widehat{\mathcal{P}}(u,v;q)$ depends polynomially on the coefficients of $q(u)$.

Lemma 7.2. *Let* $\hbar = 1$ *and* $z_1 = \dots = z_n = 0$. *Then* $\mathcal{P}_\hbar(u,v;z_1,\dots,z_n;q) = \widehat{\mathcal{P}}(u,v;q)$.

The proof is given in Section A.4.

Theorem 7.3. *The subalgebra* $\mathcal{A}_n^S$ *is generated by the elements* $G_2^{[n]},\dots,G_n^{[n]}$. *More precisely,* $T^{[n]}(u,v) = \widehat{\mathcal{P}}(u,v;S_1^{[n]})$, *where* $T^{[n]}(u,v)$ *is given by* (6.10) *with* $\hbar = 1$.

Proof. Since $G_k^{[n]} = S_{1,k-1}^{[n]}$, see (6.8), (7.3), the statement follows from Theorem 6.7 and Lemma 7.2. □

Let $\gamma_n = \sigma_{1,2}\,\sigma_{2,3}\,\dots\,\sigma_{n-1,n} = G_n^{[n]}$. Then

$$\gamma_n^{-1}\,S_1^{[n]}(u) \,=\, 1 + \sum_{k=1}^{n-2} \gamma_n^{-1}\,G_{n-k}^{[n]}\,u^k + \gamma_n^{-1}\,n\,u^{n-1} \tag{7.6}$$

and there exists the power series

$$\log\bigl(\gamma_n^{-1}\,S_1^{[n]}(u)\bigr) \,=\, \sum_{k=1}^{\infty} I_k^{[n]}\,u^k \tag{7.7}$$

with coefficients in $\mathbb{C}[\mathfrak{S}_n]$.

Recall that $\pi_{1,\dots,k}^{[n]} : \mathbb{C}[\mathfrak{S}_k] \to \mathbb{C}[\mathfrak{S}_n]$ is the embedding induced by the correspondence $i \mapsto i$. For brevity, set $\pi_k = \pi_{1,\dots,k}^{[n]}$.

Theorem 7.4. *For every* $k \in \mathbb{Z}_{>0}$ *there exists an element* $\theta_k \in \mathfrak{S}_{k+1}$ *independent of* n *such that*

$$I_k^{[n]} \,=\, \sum_{m=0}^{n-1} \gamma_n^m\,\pi_{k+1}(\theta_k)\,\gamma_n^{-m}\,, \qquad k = 1,\dots,n-2\,. \tag{7.8}$$

For example, one can take $\theta_1 = \sigma_{1,2}$, $\theta_2 = (\sigma_{2,3}\,\sigma_{1,2} - \sigma_{1,2}\,\sigma_{2,3} - 1)/2$.

Proof. The proof is by the same arguments as in the Appendix of [L].

Given integers $0 = i_0 < i_1 < \dots < i_k < i_{k+1} = n+1$ with $k < n$ define $\vec{\sigma}_{i_1,\dots,i_k} \in \mathfrak{S}_n$ as follows. Choose $m \in \{1,\dots,k\}$ such that $i_{m+1} - i_m > 1$ and set

$$\vec{\sigma}_{i_1,\dots,i_k} \,=\, \sigma_{i_m,i_m+1}\dots\sigma_{i_1,i_1+1}\;\sigma_{i_k,i_k+1}\dots\sigma_{i_{m+1},i_{m+1}+1}\,,$$

where $\sigma_{n,n+1}$ is understood as $\sigma_{1,n}$. It is easy to see that $\vec{\sigma}_{i_1,\dots,i_k}$ does not depend on the choice of m. By (7.2), we have $\gamma_n^{-1}\,G_{n-k}^{[n]} = \sum_{1\leqslant i_1<\dots<i_k\leqslant n} \vec{\sigma}_{i_1,\dots,i_k}$. Consider the series

$$\log\Biggl(1 + \sum_{k=1}^{n-2}\;\sum_{1\leqslant i_1<\dots<i_k\leqslant n} \vec{\sigma}_{i_1,\dots,i_k}\,u_{i_1}\dots u_{i_k}\Biggr) \,=\, \sum_{s_1,\dots,s_n=0}^{\infty} \varphi_{s_1,\dots,s_n}\,u_1^{s_1}\dots u_n^{s_n} \tag{7.9}$$

in the variables $u_1,\dots,u_n$ with coefficients in $\mathbb{C}[\mathfrak{S}_n]$. It is easy to see that $\varphi_{s_2,\dots,s_n,s_1} = \gamma_n\,\varphi_{s_1,\dots,s_n}\,\gamma_n^{-1}$. Moreover, for any $m < n-1$ and any $s_1,\dots,s_m$ we have that $\varphi_{s_1,\dots,s_m,0,\dots,0} \in \pi_{m+1}(\mathfrak{S}_{m+1})$ and $\pi_{m+1}^{-1}(\varphi_{s_1,\dots,s_m,0,\dots,0})$ does not depend on n.

One can check that $\varphi_{r_1,\dots,r_{n-2},0}=0$ if $r_i=0$ for some $i=2,\dots,n-3$. This can be done by determining $\varphi_{s_1,\dots,s_n}$ recursively from

$$\exp\Bigl(\sum_{s_1,\dots,s_n=0}^{\infty}\varphi_{s_1,\dots,s_n}\,u_1^{s_1}\dots u_n^{s_n}\Bigr)\;=\;1+\sum_{k=1}^{n-2}\;\sum_{1\leqslant i_1<\dots<i_k\leqslant n}\vec{\sigma}_{i_1,\dots,i_k}\,u_{i_1}\dots u_{i_k}\,,$$

see (7.9), and employing induction on $\sum_{i=1}^{n-1}r_i$. Define the elements $\theta_k\in\mathfrak{S}_{k+1}$ by

$$\pi_{k+1}(\theta_k)\;=\;\sum_{m=1}^{k}\;\sum_{\substack{s_1,\dots,s_m=1\\ s_1+\dots+s_m=k}}^{k-m+1}\varphi_{s_1,\dots,s_m,0,\dots,0}\,.$$

Since

$$\log\Bigl(1+\sum_{k=1}^{n-2}\gamma_n^{-1}\,G_{n-k}^{[n]}\,u^k\Bigr)\;=\;\sum_{k=1}^{n-2}I_k^{[n]}\,u^k+O(u^{n-1})$$

taking $u_1=\dots=u_n=u$ in (7.9) yields (7.8). $\square$

Because of formula (7.8), we call the elements $I_1^{[n]},\dots,I_{n-2}^{[n]}$ the *local charges.*

Corollary 7.5. *The subalgebra $\mathcal{A}_n^S$ is generated by γ_n and the local charges $I_1^{[n]},\dots,I_{n-2}^{[n]}$.*

Proof. The claim follows from Theorem 7.3. $\square$

Recall that $\mathfrak{S}_n$ acts on $(\mathbb{C}^N)^{\otimes n}$ by permuting the tensor factors, and $\varpi_n:\mathbb{C}[\mathfrak{S}_n]\to\mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$ is the corresponding homomorphism. Then

$$\varpi_n(I_1^{[n]})\;=\;\sum_{a=1}^{n}\sum_{i,j=1}^{N}E_{i,j}^{(a)}\otimes E_{j,i}^{(a+1)}\,,$$

where $E_{i,j}^{(a)}=1^{\otimes(a-1)}\otimes E_{i,j}\otimes 1^{\otimes(n-a)}$ and $E_{i,j}^{(n+1)}$ stands for $E_{i,j}^{(1)}$. In particular, for $N=2$ the operator $n-2\,\varpi_n(I_1^{[n]})\in\mathrm{End}\bigl((\mathbb{C}^2)^{\otimes n}\bigr)$ is the Hamiltonian of the XXX Heisenberg model.

Appendix

A.1. **Bethe subalgebra of $U(\mathfrak{gl}_N[t])$ and Bethe algebras $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.** Let $e_{i,j}$ be the standard generators of $\mathfrak{gl}_N$ satisfying the relations $[e_{i,j},e_{k,l}]=\delta_{j,k}\,e_{i,l}-\delta_{i,l}\,e_{k,j}$. Let $\mathfrak{gl}_N[t]=\mathfrak{gl}_N\otimes\mathbb{C}[t]$ be the Lie algebra of $\mathfrak{gl}_N$-valued polynomials with the pointwise commutator. We identify the Lie algebra $\mathfrak{gl}_N$ with the subalgebra $\mathfrak{gl}_N\otimes 1$ of constant polynomials in $\mathfrak{gl}_N[t]$.

Consider first-order formal differential operators in u:

$$\widetilde{X}_{i,j}\;=\;\delta_{i,j}\,\partial_u-\sum_{r=0}^{\infty}(e_{i,j}\otimes t^r)\,u^{-r-1}\,,\qquad i,j=1,\dots,N\,,$$

and the N-th order formal differential operator in u

$$\widetilde{\mathcal{D}}\;=\;\sum_{\sigma\in\mathfrak{S}_N}\mathrm{sign}(\sigma)\,\widetilde{X}_{\sigma(1),1}\,\widetilde{X}_{\sigma(2),2}\dots\widetilde{X}_{\sigma(N),N}\;=\;\partial_u^N+\sum_{i=1}^{N}\sum_{j=i}^{\infty}(-1)^i\,B_{i,j}\,u^{-j}\,\partial_u^{N-i}\,.$$

The unital subalgebra $\mathcal{B} \subset U(\mathfrak{gl}_N[t])$ generated by the elements $B_{i,j}$, $i = 1, \dots, N$, $j \in \mathbb{Z}_{\geqslant i}$, is called the *Bethe subalgebra* of $U(\mathfrak{gl}_N[t])$.

Theorem A.1 ([T], [MTV1])**.** *The subalgebra $\mathcal{B}$ is commutative and commutes with the subalgebra $U(\mathfrak{gl}_N) \subset U(\mathfrak{gl}_N[t])$.* □

Let M be a $\mathfrak{gl}_N$-module. Recall that $v \in M$ has weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ if $e_{i,i}v = \lambda_i v$ for all $i = 1, \dots, N$, and $v \in M$ is singular if $e_{i,j}v = 0$ for all $i < j$.

Given $b \in \mathbb{C}$, each $\mathfrak{gl}_N$-module M becomes the evaluation module $M(b)$ over $\mathfrak{gl}_N[t]$ via the homomorphism $\mathfrak{gl}_N[t] \to \mathfrak{gl}_N$, $g \otimes t^r \mapsto g\, b^r$ for any $g \in \mathfrak{gl}_N$, $r \in \mathbb{Z}_{ge0}$.

Let $L_{\boldsymbol{\lambda}}$ be the irreducible highest weight $\mathfrak{gl}_N$-module of highest weight $\boldsymbol{\lambda}$. For a collection $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$ of $\mathfrak{gl}_N$-weights, consider the tensor product $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}$ of $\mathfrak{gl}_N$-modules and denote by $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda}} \subset \otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}$ the subspace of singular vectors of weight $\boldsymbol{\lambda}$.

Given complex numbers $\boldsymbol{b} = (b_1, \dots, b_k)$, consider the tensor product $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)$ of evaluation $\mathfrak{gl}_N[t]$-modules. The action of $\mathcal{B}$ on $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)$ preserves the subspace $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda}}$. Let $\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ denote the corresponding $\mathcal{B}$-module. By definition, the algebras $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda}}$ and $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ are the images of $\mathcal{B}$ in $\operatorname{End}\bigl(\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)\bigr)$ and $\operatorname{End}(\mathcal{M}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}})$, respectively.

The algebras $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ were studied in [MTV4]. When $\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}$ and $\boldsymbol{\lambda}$ are partitions with at most N parts, and $b_1, \dots, b_k$ are distinct, the $\mathfrak{gl}_N[t]$-module $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)$ is irreducible. If some of $b_1, \dots, b_k$ coincide, then the $\mathfrak{gl}_N[t]$-module $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)$ is a direct sum of irreducible submodules, and these submodules are tensor products of evaluation $\mathfrak{gl}_N[t]$-modules. In such a case, the algebra $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ is isomorphic to the direct sum $\bigoplus_{\boldsymbol{\Lambda}',\boldsymbol{b}'} \mathcal{B}_{\boldsymbol{\Lambda}',\boldsymbol{\lambda},\boldsymbol{b}'}$, where the pairs $\boldsymbol{\Lambda}', \boldsymbol{b}'$ label nonequivalent irreducible $\mathfrak{gl}_N[t]$-submodules of $\otimes_{i=1}^k L_{\boldsymbol{\lambda}^{(i)}}(b_i)$.

Recall that $E_{i,j} \in \operatorname{End}(\mathbb{C}^N)$ is the matrix with only one nonzero entry equal to 1 at the intersection of the i-th row and j-th column. The assignment $e_{i,j} \mapsto E_{i,j}$ makes $\mathbb{C}^N$ into the $\mathfrak{gl}_N$-module isomorphic to $L_{\boldsymbol{\omega}}$, where $\boldsymbol{\omega} = (1, 0, \dots, 0)$ is the first fundamental weight of $\mathfrak{gl}_N$. The algebras $\mathcal{B}_{n,N}(z_1, \dots, z_n)$ and $\mathcal{B}_{n,N,\boldsymbol{\lambda}}(z_1, \dots, z_n)$, introduced in Section 3, coincide respectively with the algebras $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda}}$ and $\mathcal{B}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ for $\boldsymbol{\Lambda} = (\boldsymbol{\omega}, \dots, \boldsymbol{\omega})$ and $\boldsymbol{b} = (z_1, \dots, z_n)$.

A.2. **Algebra $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$.** Let $K_1, \dots, K_N$ and $b_1, \dots, b_k$ be two collections of complex numbers, distinct within each collection. Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)}$ be partitions with at most N parts. The algebra $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$, where $\boldsymbol{\Lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(k)})$, $\boldsymbol{b} = (b_1, \dots, b_k)$ and the dependence on $K_1, \dots, K_N$ is suppressed, was defined for these data and studied in [MTV2, Section 5].

The algebra $\mathcal{E}_{\boldsymbol{\mu}}(z_1, \dots, z_n)$, used in the proof of Theorem 4.5, coincides with the algebra $\mathcal{O}_{\boldsymbol{\Lambda},\boldsymbol{\lambda},\boldsymbol{b}}$ for the following choice of parameters: $N = n$, $k = 1$, $b_1 = 0$, $\boldsymbol{\lambda}^{(1)} = \boldsymbol{\mu}$, $\boldsymbol{\lambda} = (1, \dots, 1)$, and $K_i = z_i$ for all $i = 1, \dots, n$.

A.3. **Bethe subalgebra of the Yangian $Y(\mathfrak{gl}_N)$.** The Yangian $Y(\mathfrak{gl}_N)$ is a unital associative algebra with generators $t_{i,j}^{\{s\}}$ for $i, j = 1, \dots, N$, $s \in \mathbb{Z}_{>0}$, subject to relations

$$(u-v)\bigl[t_{i,j}(u), t_{k,l}(v)\bigr] \,=\, t_{k,j}(v)\, t_{i,l}(u) - t_{k,j}(u)\, t_{i,l}(v)\,, \qquad i, j, k, l = 1, \dots, N\,,$$

where $t_{i,j}(u) = \delta_{i,j} + \sum_{s=1}^{\infty} t_{i,j}^{\{s\}} u^{-s}$. The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct $\Delta : Y(\mathfrak{gl}_N) \to Y(\mathfrak{gl}_N) \otimes Y(\mathfrak{gl}_N)$ given by $\Delta\bigl(t_{i,j}(u)\bigr) = \sum_{k=1}^{N} t_{k,j}(u) \otimes t_{i,k}(u)$ for $i,j = 1,\dots,N$. The Yangian $Y(\mathfrak{gl}_N)$ contains $U(\mathfrak{gl}_N)$ as a Hopf subalgebra, the embedding given by $e_{i,j} \mapsto t_{j,i}^{\{1\}}$.

Set

$$\mathcal{T}_m(u) \,=\, \sum_{\sigma\in\mathfrak{S}_m} \sum_{1\leqslant i_1<\dots<i_m\leqslant n} \mathrm{sign}(\sigma)\, t_{i_{\sigma(1)},i_1}(u-m+1)\dots t_{i_{\sigma(m)},i_m}(u)\,. \tag{A.1}$$

The series $\mathcal{T}_1(u),\dots,\mathcal{T}_N(u)$, are called *transfer matrices*, see [KS] and a recent exposition in [MTV1]. Formula (A.1) is obtained from formulae (4.13) and (4.9) in [MTV1].

The subalgebra $\mathcal{B}_N^Y \subset Y(\mathfrak{gl}_N)$ generated by coefficients of the series $\mathcal{T}_1(u),\dots,\mathcal{T}_N(u)$ is called the *Bethe subalgebra* of $Y(\mathfrak{gl}_N)$.

Theorem A.2 ([KS])**.** *The subalgebra $\mathcal{B}_N^Y$ is commutative and commutes with the subalgebra $U(\mathfrak{gl}_N) \subset Y(\mathfrak{gl}_N)$.* □

Let $E_{i,j} \in \mathrm{End}(\mathbb{C}^N)$ be as in Section A.1. Given a complex number x, the assignment $t_{i,j}^{\{s\}} \mapsto E_{j,i}x^s$ for $i,j = 1,\dots,N$, $s \in \mathbb{Z}_{>0}$, that is, $t_{i,j}(u) \mapsto \delta_{i,j} + E_{j,i}(u-x)^{-1}$, makes $\mathbb{C}^N$ into the evaluation module $V(x)$ over $Y(\mathfrak{gl}_N)$.

For complex numbers $x_1,\dots,x_n$, we denote by $\psi_{x_1,\dots,x_n} : Y(\mathfrak{gl}_N) \to \mathrm{End}\bigl((\mathbb{C}^N)^{\otimes n}\bigr)$ the defining homomorphism of the $Y(\mathfrak{gl}_N)$-module $V(x_1)\otimes\dots\otimes V(x_n)$. This homomorphism is used in the proof of Theorem 5.18.

A.4. **Proof of Lemma 7.2.** In the proof, we are using several identities for rational functions whose verification is left to a reader.

Let C be the matrix with entries $C_{ab} = \prod_{c=1}^{a-1}(z_b - z_c)$ for $a \leqslant b$, and $C_{ab} = 0$ for $a > b$. Then the entries of C^{-1} are $(C^{-1})_{ab} = \prod_{c=1,\,c\neq a}^{b}(z_a - z_c)^{-1}$ for $a \leqslant b$, and $(C^{-1})_{ab} = 0$ for $a > b$.

Recall that Z and $\widehat{Z}$ are the matrices with entries $Z_{ab} = z_a\,\delta_{ab}$ and $\widehat{Z}_{ab} = \delta_{a,b-1}$. It is straightforward to see that

$$C Z C^{-1} \,=\, Z + \widehat{Z}\,. \tag{A.2}$$

Let $Q_\hbar$ be the matrix given by (6.5), (6.6), and $\widetilde{Q} = CQ_\hbar C^{-1}$ with $\hbar = 1$. Then after a minor simplification

$$\widetilde{Q}_{ab} \,=\, \sum_{c=a}^{n}\sum_{d=1}^{b} \frac{q(z_c)}{z_c - z_d + 1} \prod_{\substack{r=a\\ r\neq c}}^{n} \frac{1}{z_c - z_r} \prod_{\substack{s=1\\ s\neq d}}^{b} \frac{1}{z_d - z_s}\,.$$

Taking the sum over d, we get

$$\widetilde{Q}_{ab} \,=\, \sum_{c=a}^{n} q(z_c) \prod_{\substack{r=a\\ r\neq c}}^{n} \frac{1}{z_c - z_r} \prod_{s=1}^{b} \frac{1}{z_c - z_s + 1} \,=\, \frac{1}{2\pi i}\int_{\gamma} q(u) \prod_{r=a}^{n} \frac{1}{u - z_r} \prod_{s=1}^{b} \frac{1}{u - z_s + 1}\, du\,,$$

where γ is a simple closed curve such that the points $z_1, \dots, z_n$ are inside γ and $z_1 - 1$, $\dots, z_n - 1$ are outside. For instance, if $|z_a| < 1/3$ for all a, one can take γ to be the circle $|u| = 1/2$. Therefore, in the limit $z_a \to 0$ for all $a = 1, \dots, n$,

$$\widetilde{Q}_{ab} \to \frac{1}{2\pi i} \int\limits_{|u|=1/2} \frac{q(u)}{u^{n-a+1}(u+1)^b}\, du = \widehat{Q}_{ab}\,,$$

cf. (6.5). The last formula together with (A.2), (6.7) and (7.5) implies that

$$\mathcal{P}_\hbar(u,v;z_1,\dots,z_n;q) \to \widehat{\mathcal{P}}(u,v;q)\,, \qquad z_a \to 0\,, \quad a = 1,\dots,n\,,$$

which proves Lemma (7.2). □

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